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Effective least squares approximation method for estimating the rhythm function of cyclic random process

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Abstract

The work is devoted to a problem of the rhythm function estimation of a cyclic random process, which is based on the least squares approximation methods instead of well-known interpolation approach. Analytical dependencies between errors of estimation of a discrete rhythm function and errors of segmentation of a cyclic random process into cycles and zones were constructed. This made it possible to develop a procedure for calculating and controlling errors of estimating rhythm function of a cyclic random process as certain functions of errors of the segmentation method. The general problem of least squares approximation of the rhythm function of a cyclic random process is formulated as a problem of optimal selection of a parametric function derived from a predetermined class of functions that satisfy the necessary and sufficient conditions of the rhythm function of a cyclic random process. New parametric classes of rhythm characteristics of cyclic random processes such as parametric monomials of degree *k*, parametric logarithmic functions and parametric exponential functions have been built. The advantage of considered method over well-known interpolation approach refers to the improvement of accuracy of rhythm function estimation and reduction of the rhythm function estimation parameters' number. For example, in presented computer simulation experiment for the parametric class of monomials of degree 2, average value of the mean square errors for 500 simulations in the case of the interpolation is over 40 times higher than the corresponding value for approximation. Moreover, for that parametric class, the number of estimated parameters is almost equal to doubled number of considered cycles in the case of piecewise linear interpolation and is reduced to 1 for least square approximation. The results obtained in the work constitute the basis for improvement of rhythm-adaptive methods and spectral analysis of cyclic random processes, including the area of statistical methods for detecting hidden cyclic structures of the investigated cyclic stochastic signals with an irregular rhythm.

Keywords: Rhythm function estimation, Approximation method, Cyclic random process, Irregular rhythm

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1 Introduction

The development of digital processing methods for cyclic stochastic signals is a significant task for many fields of science and technology. In particular, cyclic stochastic signals exist in medicine (electrocardiosignals, magnetocardiosignals, sphygmocardiosignals, photoplethysmocardiosignals, phonocardiosignals, rheocardiosignals), telecommunications (amplitude, angle and phase modulation signals), power engineering (cyclic processes of electricity, gas, and oil consumption), mechanics (vibration signals on rotating mechanisms), economics (cyclic economic processes), systems for diagnosing the state of materials (cyclic processes of relief formation on the surface of metals), astrophysics (signals of pulsars and quasars). Methods of digital processing of cyclic stochastic signals include statistical estimation of probabilistic characteristics, spectral analysis, classification. And forecasting of cyclic stochastic signals. In particular, methods of digital signal processing are. The basis of modern technologies for hidden periodicity detection (more generally, for detection of hidden cyclic structure) in research of signals.

Currently, many methods of digital processing of cyclic stochastic signals are explored with regard to the problem of accuracy, computational complexity and level of interpretability. In particular, there are various model-based statistical methods and data-based processing methods involving modern machine learning technologies. Data-based methods of digital processing of cyclic stochastic signals work with inductive methods of both classical and deep machine learning with different types of neural networks [1–4]. Model-based methods of digital processing of cyclic stochastic signals use such mathematical models as a periodic Markov process (or chain) [5], random processes with independent periodic increments, linear periodic random process [6, 7], periodically correlated (cyclostationary correlated) random processes (time series) [8], periodically distributed (cyclostationary) random processes (time series) [9, 10], almost periodically correlated (almost cyclostationary correlated) random process and almost periodic (almost cyclostationary) random process [11]. In [12], mathematical models of cyclic stochastic signals were built, which generalize cyclostationary and almost cyclostationary random processes and time series in a certain way. Mathematical models and methods of processing cyclic stochastic signals with an irregular (variable) rhythm were investigated in [13–18].

One of the approaches to the modeling and processing of cyclic stochastic signals is based on the theory of cyclic random processes and vectors. This approach has been successfully applied to solve problems of digital processing of cyclic stochastic signals in medical diagnostics, in brain-computer non-invasive interfaces [19], in systems for diagnosing the state of materials [20] and in energy forecasting [21]. The main advantage of this model-based approach, unlike the existing ones, is the presence of formal and technological means of adaptation to changes occurring in the rhythm of the analyzed cyclic signals [22], as well as an integrative (holistic) approach to mathematical and computer modeling of cyclic stochastic signals with both regular (constant) and irregular (variable) rhythm [23].

The primary stage of all processing techniques (discretization, statistical estimation of probabilistic characteristics, spectral analysis) of cyclic stochastic signals, within the framework of the theory of cyclic random processes and vectors, consists of the rhythm function estimation (just like the need to estimate the period for statistical processing of cyclostationary random processes) [19–21]. The segmentation procedure of cyclic signals, i.e. separation into cycles and zones (smaller segments in each cycle), is the first step to estimate the rhythm function. A large number of methods for effective segmentation of cyclic stochastic signals have been developed, for example, methods of detection and selection of cycles and zones of electrocardiosignals [24–29]. In publications [19–21], the problem of the rhythm function estimation of a cyclic random process was solved on the basis of discrete rhythm function interpolation methods. In particular, the methods of piecewise linear, piecewise quadratic and adaptive interpolations were used [30].

However, interpolation of the discrete rhythm function has two important drawbacks: 1) high dimensionality of the parameter vector of the interpolation function; 2) it ignores the segmentation errors which influence the estimation of the discrete rhythm function. The high dimensionality of the vector of rhythm function parameters imposes high requirements for computing resources in classification tasks based on the information about changes in the rhythm of cyclic signals.

For the simplest case of the piecewise linear interpolation of the rhythm function, the number of estimation parameters is equal to the doubled product of the number of cycles and the number of zones in each considered cycle in the segmentation process. The interpolation methods do not take into account the real errors of estimation of the discrete rhythm function, although such errors always occur, since the discrete rhythm function is determined through the zone-cyclic structure of the studied cyclic random process by segmenting it. Assuming the existence of such errors, there is no need to require an exact passage of the rhythm function estimate through discrete values, because they contain errors. It is enough to require a close pass (within tolerance) to these nodal points.

Considering the above, one of the possible approaches to reduce the dimensionality of the parameter vector, which represents the estimation of the rhythm function and ensures the necessary accuracy of the estimation within the limits of permissible errors, is the development of approximation methods for estimating the rhythm function. An approximate approach to the rhythm function estimation of a cyclic random process requires structural and parametric identification of the mathematical model of the rhythm function. The procedure for such identification should be based on information about the moments of beginning of signal cycles and zones obtained by the chosen method of cyclic signal segmentation. It should take into account the errors of the segmentation method as well as information (if it exists) about the class of possible mathematical models of rhythm functions in the domain that researched cyclic stochastic signals and phenomena come from. The availability of a priori information about the class of mathematical models of rhythm functions significantly simplifies the task of structural and parametric identification, reducing it to the task of parametric identification. From a practical point of view, such information can be obtained both on the basis of visual analysis of graphs of estimates of discrete rhythm functions, and on the basis of knowledge and recommendations of a domain expert (cardiologist, neurobiologist, economist, material scientist).

In the area of cyclic random processes, the development of structural and parametric identification of the mathematical model of rhythm function requires components such as:

- Parametric classes of possible models of the rhythm function within which it is necessary to find the optimal (quasi-optimal) estimate of the rhythm function of the analyzed cyclic signal;
- A method for optimal (quasi-optimal) selection of model parameters with the appropriate rhythm function class, based on a given metric (RMS, absolute, etc.);
- A procedure for calculating and controlling the errors of rhythm function estimation of a cyclic random process as certain functions of errors of the segmentation method of this process.

The task of parametric classes construction for approximation functions that satisfy the requirements of the rhythm function is non-trivial because the conditions of the rhythm function impose a number of limitations on design of such parametric classes. In [23], a parabolic class of rhythm functions was constructed. However, this is not sufficient for many cases that may occur. Therefore, one of the important goals of the article is to build several new parametric classes of rhythm functions. The second problem considered in the article is the selection of optimal model parameters for the appropriate class of rhythm functions based on an RMS metric. This method is based on the classical least squares approximation method and should guarantee the construction of an estimate of the rhythm function, which satisfies the sufficient and necessary conditions and is consistent with the errors of segmentation method. The third problem addressed in the article is building analytical dependencies between the estimation errors of the discrete rhythm function and the errors of the cyclic random process segmentation method. As the result the procedure for calculating and controlling the errors of estimating rhythm function as certain function of errors of the segmentation method of considered random process is developed. Solving all these three problems develops the new effective approximation method for estimating rhythm function of a cyclic random process, which has significant advantages over currently used interpolation methods.

The article is organized as follows. Chapter 2 is devoted to basic information about cyclic random processes, their zone-cyclic time structure and analytical dependencies connecting the values of the discrete rhythm function through the moments of the beginnings of the cycles and zones of the cyclic random process. In Chapter 3, the analytical relationships between the estimation errors of a discrete rhythm function and the segmentation errors of a cyclic random process are investigated. Chapter 4 contains the formulation of the general problem of the rhythm function's approximation of a cyclic random process, based on the information about its zone-cyclic time structure. Chapter 5 is devoted to the formation of the system of interrelated parametric classes of rhythm characteristics of cyclic random processes. In Chapter 6, based on computer simulation experiments, the errors of least squares approximation method and piecewise linear interpolation of the discrete rhythm functions are determined and compared, as well as the number of parameters of the approximant and interpolant. Chapter 7 contains the main conclusions of the article and prospects for further research.

2 Preliminaries

2.1 The cyclic random processes as mathematical models of cyclic signals

Based on [23], we describe the fundamental properties of a cyclic random process, such as the definition of the rhythm function and a cyclic random process of continuous parameter. In general, the mathematical model of the cyclic signal is some random process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$

($\xi : \boldsymbol{R} \to L_2(\boldsymbol{\Omega}, \boldsymbol{P})$), given on the probability space $(\boldsymbol{\Omega}, \boldsymbol{F}, \boldsymbol{P})$ and on the set $\boldsymbol{R}$ of real numbers.

The argument t can have a physical interpretation of the spatial or time coordinate, and the set of values is a space of random variables given in the same probability space $(\boldsymbol{\Omega}, \boldsymbol{F}, \boldsymbol{P})$.

Definition 1 The separable random process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$ is called the cyclic random process (cyclically distributed random process) of a continuous parameter, if for it exists such a function $T(t, n), t \in \boldsymbol{R}, n \in \boldsymbol{Z}$, which satisfies the conditions (1)–(3) of the rhythm function and for any $t_1, ..., t_k$ from the set of separability of the process $\xi(\omega, t), \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}$, k-dimensional random vectors $(\xi(\omega, t_1), \ldots, \xi(\omega, t_k))$ and $(\xi(\omega, t_1 + T(t_1, n)), \ldots, \xi(\omega, t_k + T(t_k, n)))$ are stochastically equivalent for all $n \in \boldsymbol{Z}$ and for all $k \in \boldsymbol{N}$.

Note that k-dimensional random vectors are stochastically equivalent, if they are the same.

k-dimensional distribution functions.

The rhythm function $T(t, n)$ has the following properties which hold for every $t \in \boldsymbol{R}$ and $n \in \boldsymbol{Z}$ [23]:

(1)

$$\begin{cases} T(t,n) > 0 (T(t,1) < \infty), \text{for } n > 0, \\ T(t,n) = 0, \text{for } n = 0, \\ T(t,n) < 0, \text{for } n < 0, \end{cases} \tag{1}$$

(2) For any $t_1, t_2 \in \boldsymbol{R}$, where $t_1 < t_2$, a strict inequality holds:

$$T(t_1, n) + t_1 < T(t_2, n) + t_2, \tag{2}$$

(3) Function $T(t, n)$ is the smallest in modulus $\left(|T(t,n)| \leq \left|T_\gamma(t,n)\right|\right)$ among all such functions $\left\{T_\gamma(t,n), \gamma \in \boldsymbol{N}\right\}$ which satisfy (1) and (2), namely:

$$|T(t,n)| = \min_{\gamma \in \boldsymbol{N}} \left\{\left|T_\gamma(t,n)\right|, \gamma \in \boldsymbol{N}\right\} \tag{3}$$

The rhythm function $T(t, n)$ determines the law of changing the time intervals between the single-phase values of the cyclic random signals.

For k-dimensional distribution function $F_{k_\xi}(x_1,\ldots,x_k,t_1,\ldots,t_k)$ from the family of consistent distribution functions of a cyclic random process $\xi(\omega,t),\omega\in\mathbf{\Omega},t\in\boldsymbol{R}$ the following equalities are true:

$$F_{k_\xi}(x_1,\ldots,x_k,t_1,\ldots,t_k)=F_{k_\xi}(x_1,\ldots,x_k,t_1+T(t_1,n),\ldots,t_k+T(t_k,n)) \tag{4}$$

where the definition domain of the i-th zone

$$x_1,\ldots,x_k,t_1,\ldots,t_k\in R,n\in Z,k\in N$$

If $T(t,n)=n\cdot T,\ T=const,\ T>0$, then the cyclic (cyclically correlated) random process with a regular (stable, unchanging) rhythm is obtained, which in the literature is known as a cyclostationary (cyclostationary correlated) random process or periodic (periodic correlated) random process. If $T(t,n)\neq n\cdot T$, then the cyclic (cyclically correlated) random process with an irregular (variable, unstable) rhythm is considered [23].

2.2 Zone-cyclic Time Structure of a Cyclic Random Process

Let us introduce notations for $m\in\boldsymbol{Z},d_m\in\boldsymbol{R}(0<d_{m+1}-d_m<\infty)$ [23]:

the definition domain of the m-th cycle of cyclic random process $\xi(\omega,t)$:

$$\boldsymbol{W}_{c_m}=[d_m,d_{m+1}),$$

where $\bigcup_{m\in\boldsymbol{Z}}\boldsymbol{W}_{c_m}=\boldsymbol{R}$,

the set of the beginnings of cycles of a cyclic random process:

$$\boldsymbol{D}_c=\left\{d_m,m\in\boldsymbol{Z}\right\}$$

Let us notice the following relations:

$$W_{c_m}\neq\emptyset,W_{c_{m_1}}\cap W_{c_{m_2}}=\emptyset,m_1\neq m_2,m,m_1,m_2\in Z \tag{5}$$

Based on the work [23], the cyclic random process can be represented in the form of the following construction:

$$\xi(\omega,t)=\sum_{m\in\boldsymbol{Z}}\xi_m(\omega,t),\ \omega\in\boldsymbol{\Omega},\ t\in\boldsymbol{R}, \tag{6}$$

where $\xi_m(\omega,t)$- random process, which is defined as follows

$$\xi_m(\omega,t)=\xi(\omega,t)\cdot I_{\boldsymbol{W}_{c_m}}(t),\ m\in\boldsymbol{Z},\ \omega\in\boldsymbol{\Omega},\ t\in\boldsymbol{R}, \tag{7}$$

and function $I_{\boldsymbol{W}_{c_m}}(t)$ is an indicator function of the m-th cycle of the cyclic random process $\xi(\omega,t)$, which is equal to:

$$I_{\boldsymbol{W}_{c_m}}(t)=\begin{cases}1,\ t\in\boldsymbol{W}_{c_m},\\0,\ t\notin\boldsymbol{W}_{c_m}.\end{cases} \tag{8}$$

The truncation of the random process $\xi_m(\omega, t)$ to the area $\boldsymbol{W}_{c_m}$ is the m-th cycle of the cyclic random process $\xi(\omega, t)$.

Let us introduce notations for $m \in \boldsymbol{Z}, d_{m,i} \in \boldsymbol{R}, i = \overline{1, N}$:

- the definition domain of the i-th zone in the m-th cycle of cyclic random process $\xi(\omega, t)$:

$$\boldsymbol{W}_{m,i} = [d_{m,i}, d_{m,i+1}),$$

where $\bigcup_{m \in \boldsymbol{Z}} \bigcup_{i=1}^{N} \boldsymbol{W}_{m,i} = \boldsymbol{R}$,

- the set of beginnings of i-th zones in the m-th cycle of a cyclic random process:

$$\boldsymbol{D}_z = \left\{ d_{m,i}, m \in \boldsymbol{Z}, i = \overline{1, N} \right\}.$$

The following relations exist between the sets $\boldsymbol{W}_{m,i}$:

$$W_{m,i} \neq \emptyset, W_{m_1,i_1} \cap W_{m_2,i_2} = \emptyset, i_1 \neq i_2 m, m_1, m_2 \in Z, i, i_1, i_2 = 1, N \tag{9}$$

If there is a known zone structure (N zones on cycle) of cyclic random process, then the cyclic random process $\xi(\omega, t)$ can be represented in the form of the following construction:

$$\begin{aligned} \xi(\omega, t) &= \sum_{m \in \boldsymbol{Z}} \sum_{i=1}^{N} \xi_{m,i}(\omega, t), \ \omega \in \boldsymbol{\Omega}, \ t \in \boldsymbol{R}, \xi_{m,i}(\omega, t) \\ &= \xi(\omega, t) \cdot I_{\boldsymbol{W}_{m,i}}(t), \ m \in \boldsymbol{Z}, i = \overline{1, N}, \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}, \end{aligned} \tag{10}$$

where $\xi_{m,i}(\omega, t)$—random process, which is defined as follows.

$$\xi_{m,i}(\omega, t) = \xi(\omega, t) \cdot I_{\boldsymbol{W}_{m,i}}(t), \ m \in \boldsymbol{Z}, i = \overline{1, N}, \omega \in \boldsymbol{\Omega}, t \in \boldsymbol{R}, \tag{11}$$

And function $I_{\boldsymbol{W}_{m,i}}(t)$ is an indicator function of the i-th zone in the m-th cycle of cyclic random process $\xi(\omega, t)$, which is equal to:

$$I_{\boldsymbol{W}_{m,i}}(t) = \begin{cases} 1, \ t \in \boldsymbol{W}_{m,i}, \\ 0, \ t \notin \boldsymbol{W}_{m,i}. \end{cases} \tag{12}$$

The moments $d_{m,1}$ of the beginnings of the 1-th zones in the m-th cycles of cyclic random process $\xi(\omega, t)$ are equal to the moments d_m of beginnings of the m-th cycles. If the zone structure of a cyclic random process consists of only one zone $N = 1$, then $\boldsymbol{D}_z = \left\{ d_{m,1}, m \in \boldsymbol{Z} \right\} = \boldsymbol{D}_c = \left\{ d_m, m \in \boldsymbol{Z} \right\}$, that is, the zone is a cycle, and $d_{m,1} = d_m$. In the case when $N \geq 2$, the set $\boldsymbol{D}_c$ is subset of $\boldsymbol{D}_z$ and there is equality

$$\bigcup_{i=1}^{N} \boldsymbol{W}_{m,i} = \boldsymbol{W}_m \tag{13}$$

2.3 Analytical dependencies connecting the values of the discrete rhythm function throug the moments of the beginnings of the cycles and zones of the cyclic random process

As shown in [30], knowing the moments of beginnings of cycles of a cyclic random process, namely, knowing the set $\boldsymbol{D}_c = \left\{ d_m, m \in \boldsymbol{Z} \right\}$, the discrete rhythm function can be determined according to the formula

$$T(d_m, n) = d_{m+n} - d_m, d_m \in \boldsymbol{D}_c, m, n \in \boldsymbol{Z} \quad (14)$$

Also [30], knowing the moments of beginnings of the i-th zones in the m-th cycles of cyclic random process, namely, knowing the set $\boldsymbol{D}_z = \left\{d_{m,i}, m \in \boldsymbol{Z}, i = \overline{1, N}\right\}$, the discrete rhythm function can be determined according to the formula

$$T\left(d_{m,i}, n\right) = d_{m+n,i} - d_{m,i}, \text{ where } d_{m,i} \in \boldsymbol{D}_z, m, n \in \boldsymbol{Z}, i = \overline{1, N} \quad (15)$$

The methods of piecewise linear interpolation, piecewise parabolic interpolation and the method of adaptive interpolation of the rhythm function of a cyclic random process, as developed in work [30], are based on analytical Eqs. (14) and (15). Namely, the estimate $\widehat{T}(t, 1)$ of the rhythm function $T(t, n)$ is obtained as a result of interpolation of the values of the discrete rhythm functions $T(d_m, n)$ and $T\left(d_{m,i}, n\right)$.

Based on the ideas and analytical dependencies described above, Fig. 1 presents a conventional graphical representation of the deterministic cyclic function $\xi(t)$ (a degenerate case of a cyclic random process $\xi(\omega, t)$), the moments of beginnings of its cycles d_m, as well as its rhythm function $T(t, 1)$.

3 The estimation errors of discrete rhythm function of cyclic random process

Let the set of estimated beginnings of cycles $\widetilde{\boldsymbol{D}}_c = \left\{\widetilde{d}_m, m \in \boldsymbol{Z}\right\}$ or beginnings of zones in each cycle $\widetilde{\boldsymbol{D}}_z = \left\{\widetilde{d}_{m,i}, m \in \boldsymbol{Z}, i = \overline{1, N}\right\}$ be known. On the basis of equalities (14) and (15), the values of estimates $\widetilde{T}\left(\widetilde{d}_m, n\right)$ or $\widetilde{T}\left(\widetilde{d}_{m,i}, n\right)$ of the discrete rhythm functions $T(d_m, n)$ or $T\left(d_{m,i}, n\right)$ are obtained, namely:

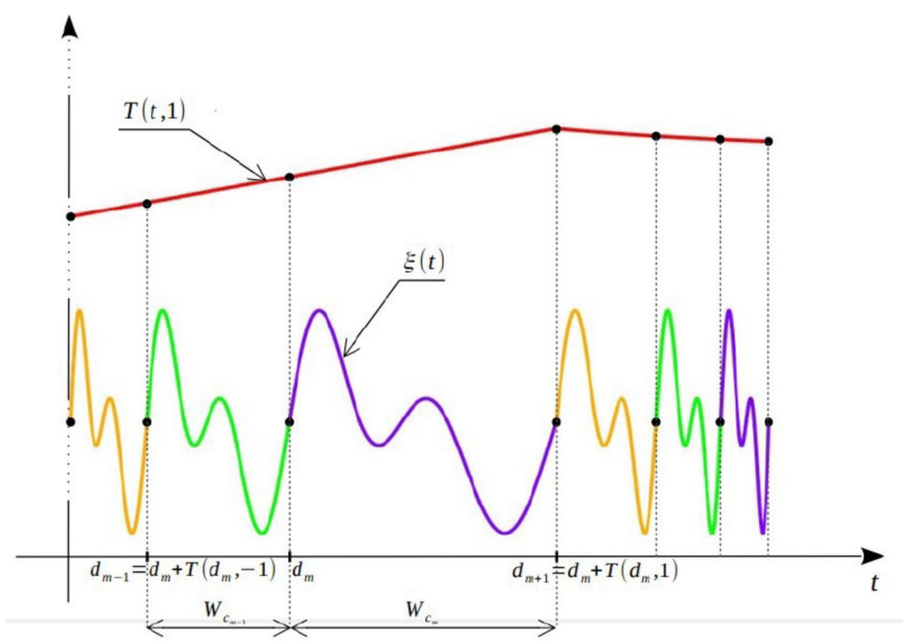

Fig. 1 Conventional graphical representation of a segment of the deterministic cyclic function $\xi(t)$, and the moments of beginnings of its cycles d_m, as well as its rhythm function $T(t, 1)$

$$\tilde{T}\left(\tilde{d}_m, n\right) = \tilde{d}_{m+n} - \tilde{d}_m, \tilde{d}_m \in \tilde{\boldsymbol{D}}_c, m, n \in \boldsymbol{Z}, \tag{16}$$

$$\tilde{T}\left(\tilde{d}_{m,i}, n\right) = \tilde{d}_{m+n,i} - \tilde{d}_{m,i}, \tilde{d}_{m,i} \in \tilde{\boldsymbol{D}}_z, m, n \in \boldsymbol{Z}, i = \overline{1, N} \tag{17}$$

Based on formulas (16) and (17), the calculation errors occurring in estimation of discrete rhythm functions $\widetilde{T}\left(\widetilde{d}_m, n\right)$ and $\widetilde{T}\left(\widetilde{d}_{m,i}, n\right)$

$$\Delta\tilde{T}\left(\tilde{d}_m, n\right) = \tilde{T}\left(\tilde{d}_m, n\right) - T\left(\tilde{d}_m, n\right), m \in \boldsymbol{Z} \tag{18}$$

$$\Delta\tilde{T}\left(\tilde{d}_{m,i}, n\right) = \tilde{T}\left(\tilde{d}_{m,i}, n\right) - T\left(\tilde{d}_{m,i}, n\right), m \in \boldsymbol{Z}, i = \overline{1, N} \tag{19}$$

can be obtained on the basis of the information about $\Delta\widetilde{d}_m$, $\Delta\widetilde{d}_{m,i}$, which are the errors connected with the beginnings of cycles and the beginnings of zones in each cycle correspondingly. For the investigated signal, the errors are expressed as:

$$\Delta\tilde{d}_m = \tilde{d}_m - d_m, m \in \boldsymbol{Z}, \tag{20}$$

$$\Delta\tilde{d}_{m,i} = \tilde{d}_{m,i} - d_{m,i}, m \in \boldsymbol{Z}, i = \overline{1, N}\,. \tag{21}$$

In (20) and (21) the $d_{m,i}$ and d_m are exact values, which are either directly specified in computer simulation experiments or determined by an expert in the considered domain (for example, a cardiologist in ECG segmentation tasks). Of course, from a practical point of view, when automatic methods of cyclic signal segmentation are used, the set of estimation errors.

$\Delta\widetilde{\boldsymbol{D}}_c = \left\{\Delta\widetilde{d}_m = \widetilde{d}_m - d_m, m \in \boldsymbol{Z}\right\}$ of the beginnings of cycles or the set of estimation errors.

$\Delta\widetilde{\boldsymbol{D}}_z = \left\{\Delta\widetilde{d}_{m,i} = \widetilde{d}_{m,i} - d_{m,i}, m \in \boldsymbol{Z}, i = \overline{1, N}\right\}$ of the beginnings of zones in each cycle are unknown, because the sets $\boldsymbol{D}_c = \left\{d_m, m \in \boldsymbol{Z}\right\}$ and $\boldsymbol{D}_z = \left\{d_{m,i}, m \in \boldsymbol{Z}, i = \overline{1, N}\right\}$ are unknown. In general, errors $\Delta\widetilde{d}_m$ and $\Delta\widetilde{d}_{m,i}$ are random variables which are described by statistical characteristics.

A more appropriate approach consists in the estimation of errors occurring in the estimation of discrete rhythm functions $\widetilde{T}\left(\widetilde{d}_m, n\right)$ and $\widetilde{T}\left(\widetilde{d}_{m,i}, n\right)$ based on the maximum absolute errors

$$\Delta_{max}\tilde{d}_* = \max_{m \in \boldsymbol{Z}} \left|\Delta\tilde{d}_m\right| = \max_{m \in \boldsymbol{Z}} \left|\tilde{d}_m - d_m\right| \tag{22}$$

$$\Delta_{max}\tilde{d}_{*,i} = \max_{m \in \boldsymbol{Z}} \left|\Delta\tilde{d}_{m,i}\right| = \max_{m \in \boldsymbol{Z}} \left|\tilde{d}_{m,i} - d_{m,i}\right|, i = \overline{1, N} \tag{23}$$

or based $\Delta_{max}\tilde{d}_* = \max_{m \in \boldsymbol{Z}} \left|\Delta\tilde{d}_m\right| = \max_{m \in \boldsymbol{Z}} \left|\tilde{d}_m - d_m\right|$ on standard deviations $\sigma_{\widetilde{d}_*}$ and $\sigma_{\widetilde{d}_{*,i}}$ which mainly characterize the accuracy of the segmentation methods of cyclic signals.

For estimates of standard deviations, the following convergence ratios hold

$$\boldsymbol{P}\left\{\lim_{M\to\infty}\sqrt{\frac{\sum_{m=-M}^{M}\left(\tilde{d}_m-d_m\right)^2}{2M}}=\sigma_{\tilde{d}_*}\right\}=1, \tag{24}$$

$$\boldsymbol{P}\left\{\lim_{M\to\infty}\sqrt{\frac{\sum_{m=-M}^{M}\left(\tilde{d}_{m,i}-d_{m,i}\right)^2}{2M}}=\sigma_{\tilde{d}_{*,i}}\right\}=1, i=\overline{1,N} \tag{25}$$

where $\boldsymbol{P}\{\cdot\}$ denotes the probability of an event.

According to the probabilistic approach, namely, from the point of view of interval estimation, the exact values of the beginnings of cycles and zones belong to the intervals with the length equal to doubled maximum absolute errors with the probability of 1. Thus:

$$\boldsymbol{P}\left\{d_m\in\left[\tilde{d}_m-\Delta_{max}\tilde{d}_*,\tilde{d}_m+\Delta_{max}\tilde{d}_*\right]\right\}=1, m\in\boldsymbol{Z}, \tag{26}$$

$$\boldsymbol{P}\left\{d_{m,i}\in\left[\tilde{d}_{m,i}-\Delta_{max}\tilde{d}_{*,i},\tilde{d}_{m,i}+\Delta_{max}\tilde{d}_{*,i}\right]\right\}=1, m\in\boldsymbol{Z}, i=\overline{1,N} \tag{27}$$

Assuming a normal distribution of $\tilde{d}_m$ and $\tilde{d}_{m,i}$, the intervals with the length equal to six standard deviations contain the exact values of the time points of the beginnings of cycles d_m and zones $d_{m,i}$ with a probability of 0.997, namely:

$$\boldsymbol{P}\left\{d_m\in\left[\tilde{d}_m-3\cdot\sigma_{\tilde{d}_*},\tilde{d}_m+3\cdot\sigma_{\tilde{d}_*}\right]\right\}\approx 0.997, m\in\boldsymbol{Z} \tag{28}$$

$$\boldsymbol{P}\left\{d_{m,i}\in\left[\tilde{d}_{m,i}-3\cdot\sigma_{\tilde{d}_{*,i}},\tilde{d}_{m,i}+3\cdot\sigma_{\tilde{d}_{*,i}}\right]\right\}\approx 0.997, m\in\boldsymbol{Z}, i=\overline{1,N} \tag{29}$$

According to the formulas (16) and (17), if maximum absolute errors are used to estimate the accuracy of cyclic random process segmentation methods, then maximum absolute errors $\Delta_{max}\tilde{T}\left(\tilde{d}_m,n\right)$ and $\Delta_{max}\tilde{T}\left(\tilde{d}_{m,i},n\right)$ of estimations $\tilde{T}\left(\tilde{d}_m,n\right)$ and $\tilde{T}\left(\tilde{d}_{m,i},n\right)$ of discrete rhythm functions $T(d_m,n)$ and $T\left(d_{m,i},n\right)$ are calculated on the bases of such dependencies:

$$\Delta_{max}\tilde{T}\left(\tilde{d}_m,n\right)=2\cdot\Delta_{max}\tilde{d}_*, \tilde{d}_m\in\tilde{\boldsymbol{D}}_c, m,n\in\boldsymbol{Z}, \tag{30}$$

$$\Delta_{max}\tilde{T}\left(\tilde{d}_{m,i},n\right)=2\cdot\Delta_{max}\tilde{d}_{*,i}, \tilde{d}_{m,i}\in\tilde{\boldsymbol{D}}_z, m,n\in\boldsymbol{Z}, i=\overline{1,N} \tag{31}$$

From the probabilistic point of view, the confidence intervals for the values of the discrete rhythm functions $T(d_m,n)$ and $T\left(d_{m,i},n\right)$ have the form:

$$\boldsymbol{P}\left\{T(d_m,n)\in\left[\tilde{T}\left(\tilde{d}_m,n\right)-2\cdot\Delta_{max}\tilde{d}_*,\tilde{T}\left(\tilde{d}_m,n\right)+2\cdot\Delta_{max}\tilde{d}_*\right]\right\}=1, m\in\boldsymbol{Z} \tag{32}$$

$$\boldsymbol{P}\left\{T\left(d_{m,i},n\right)\in\left[\tilde{T}\left(\tilde{d}_{m,i},n\right)-2\cdot\Delta_{max}\tilde{d}_{*,i},\tilde{T}\left(\tilde{d}_{m,i},n\right)+2\cdot\Delta_{max}\tilde{d}_{*,i}\right]\right\}=1,m\in\boldsymbol{Z},i=\overline{1,N} \quad (33)$$

Taking into account formulas (16) and (17), expressions (32) and (33) can be rewritten to:

$$\boldsymbol{P}\left\{T(d_m,n)\in\left[\tilde{d}_{m+n}-\tilde{d}_m-2\cdot\Delta_{max}\tilde{d}_*,\tilde{d}_{m+n}-\tilde{d}_m+2\cdot\Delta_{max}\tilde{d}_*\right]\right\}=1, \quad (34)$$

$$\boldsymbol{P}\left\{T\left(d_{m,i},n\right)\in\left[\tilde{d}_{m+n,i}-\tilde{d}_{m,i}-2\cdot\Delta_{max}\tilde{d}_{*,i},\tilde{d}_{m+n,i}-\tilde{d}_{m,i}+2\cdot\Delta_{max}\tilde{d}_{*,i}\right]\right\}=1, \quad (35)$$

where $m\in\boldsymbol{Z},i=\overline{1,N}$.

Considering formulas (16) and (17), if standard deviations are used to evaluate the accuracy of cyclic random process segmentation methods, then standard deviations $\sigma_{\tilde{T}\left(\tilde{d}_*,n\right)}$ and $\sigma_{\tilde{T}\left(\tilde{d}_{*,i},n\right)}$ related to estimations $\tilde{T}\left(\tilde{d}_m,n\right)$ and $\tilde{T}\left(\tilde{d}_{m,i},n\right)$ of discrete rhythm functions $T(d_m,n)$ and $T\left(d_{m,i},n\right)$ are calculated on the bases of such dependencies:

$$\sigma_{\tilde{T}\left(\tilde{d}_*,n\right)}=\sqrt{2}\cdot\sigma_{\tilde{d}_*},\tilde{d}_m\in\tilde{\boldsymbol{D}}_c,m,n\in\boldsymbol{Z}, \quad (36)$$

$$\sigma_{\tilde{T}\left(\tilde{d}_{*,i},n\right)}=\sqrt{2}\cdot\sigma_{\tilde{d}_{*,i}},\tilde{d}_{m,i}\in\tilde{\boldsymbol{D}}_z,m,n\in\boldsymbol{Z},i=\overline{1,N} \quad (37)$$

The confidence intervals for the values of the rhythm functions $T(d_m,n)$ and $T\left(d_{m,i},n\right)$ have the form:

$$\boldsymbol{P}\left\{T(d_m,n)\in\left[\tilde{T}\left(\tilde{d}_m,n\right)-3\sqrt{2}\cdot\sigma_{\tilde{d}_*},\tilde{T}\left(\tilde{d}_m,n\right)+3\sqrt{2}\cdot\sigma_{\tilde{d}_*}\right]\right\}\approx 0.997, \quad (38)$$

$$\boldsymbol{P}\left\{T\left(d_{m,i},n\right)\in\left[\tilde{T}\left(\tilde{d}_{m,i},n\right)-3\sqrt{2}\cdot\sigma_{\tilde{d}_{*,i}},\tilde{T}\left(\tilde{d}_{m,i},n\right)+3\sqrt{2}\cdot\sigma_{\tilde{d}_{*,i}}\right]\right\}\approx 0.997, \quad (39)$$

where $m\in\boldsymbol{Z},i=\overline{1,N}$.

Taking into account formulas (16) and (17), expressions (38) and (39) can be rewritten to:

$$\boldsymbol{P}\left\{T(d_m,n)\in\left[\tilde{d}_{m+n}-\tilde{d}_m-3\sqrt{2}\cdot\sigma_{\tilde{d}_*},\tilde{d}_{m+n}-\tilde{d}_m+3\sqrt{2}\cdot\sigma_{\tilde{d}_*}\right]\right\}\approx 0.997 \quad (40)$$

$$\boldsymbol{P}\left\{T\left(d_{m,i},n\right)\in\left[\tilde{d}_{m+n,i}-\tilde{d}_{m,i}-3\sqrt{2}\cdot\sigma_{\tilde{d}_{*,i}},\tilde{d}_{m+n,i}-\tilde{d}_{m,i}+3\sqrt{2}\cdot\sigma_{\tilde{d}_{*,i}}\right]\right\}\approx 0.997 \quad (41)$$

where $m\in\boldsymbol{Z},i=\overline{1,N}$.

If the segmentation results of cyclic signals, namely, the set of estimated beginnings of cycles.

$\tilde{\boldsymbol{D}}_c$ or set of estimated beginnings of zones in each cycle $\tilde{\boldsymbol{D}}_z$ and their errors $\Delta_{max}\tilde{d}_*$, $\Delta_{max}\tilde{d}_{*,i}$ or $\sigma_{\tilde{d}_*}$, $\sigma_{\tilde{d}_{*,i}}$ are known, then confidence intervals for the values of discrete rhythm functions $T(d_m, n)$ and $T\left(d_{m,i}, n\right)$ can be constructed on the bases of expressions (34), (35), (40) and (41).

4 The problem of least squares approximation of the rhythm function of a cyclic random process based on information about its zone-cyclic time structure

Let us formulate the general problem of the rhythm function's approximation of a cyclic random process, based on information about its zone-cyclic time structure. According to automated segmentation methods of cyclic signals, it is always advisable to consider a cyclic random process on a finite time interval, which includes a finite number of registered cycles M. Let us assume that the set of estimated beginnings of cycles $\widetilde{\boldsymbol{D}}_c = \left\{\widetilde{d}_m, m = \overline{1, M}\right\}$ or the set of estimated beginnings of zones in each cycle $\widetilde{\boldsymbol{D}}_z = \left\{\widetilde{d}_{m,i}, m = \overline{1, M}, i = \overline{1, N}\right\}$ is known. It is also assumed that standard deviations $\sigma_{\widetilde{d}_*}$ or $\sigma_{\widetilde{d}_{*,i}}$, which are used to estimate the errors of the cyclic signal segmentation method, are known.

It is worth noting that the set $\widetilde{\boldsymbol{D}}_c = \left\{\widetilde{d}_m, m = \overline{1, M}\right\}$ is nested in the set $\widetilde{\boldsymbol{D}}_z = \left\{\widetilde{d}_{m,i}, m = \overline{1, M}, i = \overline{1, N}\right\}$ (i.e. $\widetilde{\boldsymbol{D}}_c \subset \widetilde{\boldsymbol{D}}_z$). Therefore it is possible to determine the estimates of the elements of the set $\boldsymbol{D}_c$ and related errors if the elements of the set $\widetilde{\boldsymbol{D}}_z$ and their errors are known. Therefore, it is assumed that the set.

$\widetilde{\boldsymbol{D}}_z = \left\{\widetilde{d}_{m,i}, m = \overline{1, M}, i = \overline{1, N}\right\}$ and corresponding standard deviations $\sigma_{\widetilde{d}_{*,i}}$ (or standard deviations $\sigma_{\widetilde{T}\left(\widetilde{d}_{*,i}, n\right)}$) are given.

In order to estimate rhythm function, the least squares approximation method will be applied, which is consistent with the error description of cyclic signal segmentation methods, based on standard deviations $\sigma_{\widetilde{d}_{*,i}}, \sigma_{\widetilde{T}\left(\widetilde{d}_{m,i}, n\right)}$.

Let us assume that there is given a set

$$\boldsymbol{T} = \left\{T_{a_1, a_2, \ldots, a_K}(t, n) : \left(a_k, k = \overline{1, K}\right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K\right\} \tag{42}$$

which specifies the class of approximation functions satisfying the requirements of the rhythm function (1)–(3). The choice of the parametric class's type (42) can be carried out on the basis of a visual analysis of graphs of estimated discrete rhythm functions and/or recommendations of an expert from a certain field, where the analyzed cyclic signals come from (cardiologist, neurobiologist, economist, material scientist). The process of parametric class selection should take into account the number of parameters appearing in the model, namely, dimension of the parameter vector $\left(a_k, k = \overline{1, K}\right)$ for approximation $T_{a_1, a_2, \ldots, a_K}(t, 1)$ of rhythm function should be as small as possible.

Let us notice that $T(t, n)$ can be determined for any $n > 1$, on the basis of dependence

$$T(t, n) = T(t, n-1) + T(t + T(t, n-1), 1), t \in \boldsymbol{R}, \tag{43}$$

when $T(t,1)$ is known [23] (it is enough to estimate only the rhythm function $T(t,1)$ for $n=1$).

According to the above notices, the task of $T(t,1)$ estimation is formulated on the bases of the least squares approximation method, namely, the function

$$\widehat{T}(t,1) = T_{a_1,a_2,\ldots,a_K}(t,1), t \in \bigcup_{m=1}^{M}\bigcup_{i=1}^{N}\left[\widetilde{d}_{m,i},\widetilde{d}_{m,i+1}\right), \widetilde{d}_{m,i} \in \widetilde{\boldsymbol{D}}_z \tag{44}$$

is constructed by approximation of the estimation $\widetilde{T}\left(\widetilde{d}_{m,i},1\right)$, which satisfy requirements:

- the approximation function $\widehat{T}(t,1)$ must satisfy the rhythm function conditions (1)–(3);
- the approximation error $\widehat{\rho}_{\widehat{T}(t,1)}$ should be minimal in the sense of the minimum of mean-square metric between the approximant $\widehat{T}(t,1)$ and estimation $\widetilde{T}\left(\widetilde{d}_{m,i},1\right)$, namely:

$$\widehat{\rho}_{\widehat{T}(t,1)} = \widehat{\rho}_{T_{a_1,a_2,\ldots,a_K}(t,1)} = \min_{\left(a_k,k=\overline{1,K}\right)\in \boldsymbol{A}} \frac{1}{M\cdot N}\sum_{m=1}^{M}\sum_{i=1}^{N}\left(\widetilde{T}\left(\widetilde{d}_{m,i},1\right) - \widehat{T}\left(\widetilde{d}_{m,i},1\right)\right)^2 \tag{45}$$

In order to evaluate the quality of the approximation procedure, it is necessary to compare the error $\widehat{\rho}_{\widehat{T}(t,1)}$ with all standard deviations $\sigma_{\widetilde{T}\left(\widetilde{d}_{*,i},n\right)}, i=\overline{1,N}$. If the approximation error $\widehat{\rho}_{\widehat{T}(t,1)}$ does not exceed the smallest possible value $\sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)}, i=\overline{1,N}$, namely

$$\widehat{\rho}_{\widehat{T}(t,1)} \le \sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)}, \forall i=\overline{1,N} \tag{46}$$

then the obtained approximation $\widehat{T}(t,1) = T_{a_1,a_2,\ldots,a_K}(t,1)$ of the rhythm function is consistent with the accuracy of segmentation methods of the studied cyclic signal. However, quite often in practical situations this requirement is too strong and can be relaxed by checking one of the following conditions:

$$\widehat{\rho}_{\widehat{T}(t,1)} \le \max_{i=\overline{1,N}} \sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)} \tag{47}$$

$$\widehat{\rho}_{\widehat{T}(t,1)} \le \frac{1}{N}\cdot\sum_{i=1}^{N}\sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)} \tag{48}$$

Thus, the problem of approximation of the rhythm function of a cyclic random process is formulated as a problem of identification of parameters appearing in the chosen mathematical model. It consists of the choice of the optimal representation from a predetermined parametric class of functions that satisfy the necessary and sufficient conditions of the rhythm function of a cyclic random process.

Table 1 The main analytical dependencies between the characteristics of the rhythm of cyclic random processes of the continuous parameter [22]

	Rhythm function $\boldsymbol{T(t,n)}$	**Instantaneous angular frequency** $\boldsymbol{v(t)}$	**Scale transformation function** $\boldsymbol{y^{-1}(t)}$	**Dynamic shift function** $\boldsymbol{s^{-1}(t)}$
Rhythm function $T(t,n)$	$T(t,n) = T(t,n)$	$\int_{t}^{t+T(t,n)} v(t)dt = 2\pi n$	$y^{-1}(t + T(t,n)) - -y^{-1}(t) = 2\pi n$	$T(t,n) - 2\pi n =$ $= s^{-1}(t) - s^{-1}(t + T(t,n))$
Instantaneous angular frequency $v(t)$	$\int_{t}^{t+T(t,n)} v(t)dt = 2\pi n$	$v(t) = v(t)$	$v(t) = \frac{dy^{-1}(t)}{dt}$	$v(t) = \frac{ds^{-1}(t)}{dt} + 1$
Scale transformation function $y^{-1}(t)$	$y^{-1}(t + T(t,n)) - y^{-1}(t) = 2\pi n$	$y^{-1}(t) =$ $= \int_{-\infty}^{t} v(t)dt$	$y^{-1}(t) = y^{-1}(t)$	$y^{-1}(t) = t + s^{-1}(t)$
Dynamic shift function $s^{-1}(t)$	$T(t,n) - 2\pi n = s^{-1}(t) - s^{-1}(t + T(t,n))$	$s^{-1}(t) =$ $= \int_{-\infty}^{t} v(t)dt - t$	$s^{-1}(t) = y^{-1}(t) - t$	$s^{-1}(t) = s^{-1}(t)$

Table 2 The main analytical dependencies between the deviation of the rhythm function, the deviation of the instantaneous angular frequency and other characteristics of the rhythm of cyclic random processes of a continuous parameter [22]

	Deviation of the rhythm function $\varepsilon_T(t,n)$	**Deviation of the instantaneous angular frequency** $\varepsilon_\nu(t)$
Rhythm function $T(t,n)$	$\varepsilon_T(t,n) = T(t,n) - 2\pi n$	$\int_t^{t+T(t,n)} (\varepsilon_\nu(t)+1)dt = 2\pi n$
Instantaneous angular frequency $\nu(t)$	$\int_t^{t+\varepsilon_T(t,n)+2\pi n} \nu(t)dt = 2\pi n$	$\varepsilon_\nu(t) = \nu(t) - 1$
Scale transformation function $y^{-1}(t)$	$y^{-1}(t+\varepsilon_T(t,n)+2\pi n) - y^{-1}(t) = 2\pi n$	$\varepsilon_\nu(t) = \frac{dy^{-1}(t)}{dt} - 1$
Dynamic shift function $s^{-1}(t)$	$\varepsilon_T(t,n) = s^{-1}(t) - s^{-1}(t+\varepsilon_T(t,n)+2\pi n)$	$\varepsilon_\nu(t) = \frac{ds^{-1}(t)}{dt}$

5 A System of interrelated parametric classes of rhythm characteristics of cyclic random processes

5.1 Approach to construction interrelated parametric classes of rhythm characteristics of cyclic random processes

Least squares approximation of the rhythm function requires knowledge about the parametric class of approximation functions. In general, this task is quite difficult. One approach of solving this problem is based on the use of analytical dependencies among the main characteristics of the rhythm function of cyclic random processes [22]. Analytical dependencies among characteristics of the rhythm function are presented in Tables 1 and 2.

Analysis of the dependences shown in Tables 1 and 2 leads to conclusion that knowledge of the class of scale transformation and rhythm functions is sufficient to construct the corresponding parametric classes of dynamic shift, instantaneous angular frequencies, deviations of the rhythm function and deviations of the instantaneous angular frequency functions. Each of the six parametric functions depend on the same vector of parameters $\left(a_k, k=\overline{1,K}\right)$. Thus the following bijectively connected parametric functions (bijective connection is determined by a common vector of parameters for all mentioned rhythm characteristics) can be represented as parametric classes of:

scale transformation functions

$$\boldsymbol{Y}^{-1} = \left\{y^{-1}_{a_1,a_2,\ldots,a_K}(t) : \left(a_k, k=\overline{1,K}\right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K\right\} \tag{49}$$

rhythm functions (42):

$$\boldsymbol{T} = \left\{T_{a_1,a_2,\ldots,a_K}(t,n) : \left(a_k, k=\overline{1,K}\right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K\right\}$$

Dynamic shift functions

$$\boldsymbol{S}^{-1} = \left\{s^{-1}_{a_1,a_2,\ldots,a_K}(t) : \left(a_k, k=\overline{1,K}\right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K\right\} \tag{50}$$

instantaneous angular frequencies

$$\boldsymbol{V} = \left\{ v_{a_1,a_2,\ldots,a_K}(t) : \left(a_k, k = \overline{1,K} \right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K \right\} \tag{51}$$

deviations of instantaneous angular frequencies

$$\mathbf{E}_{\boldsymbol{v}} = \left\{ \varepsilon_{v\,a_1,a_2,\ldots,a_K}(t) : \left(a_k, k = \overline{1,K} \right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K \right\} \tag{52}$$

deviations of rhythm functions

$$\mathbf{E}_{\boldsymbol{T}} = \left\{ \varepsilon_{T\,a_1,a_2,\ldots,a_K}(t,n) : \left(a_k, k = \overline{1,K} \right) \in \boldsymbol{A} \subseteq \boldsymbol{R}^K \right\} \tag{53}$$

The following bijection among mentioned parametric classes of functions can be stated:

$$\boldsymbol{Y}^{-1} \iff \boldsymbol{T} \iff \boldsymbol{S}^{-1} \iff \boldsymbol{V} \iff \mathbf{E}_{\boldsymbol{v}} \iff \mathbf{E}_{\boldsymbol{T}} \tag{53}$$

In this case, a system of bijectively related parametric classes of functions can be constructed as follows.

$$\boldsymbol{SRC} = \langle \left\{ \boldsymbol{Y}^{-1}, \boldsymbol{T}, \boldsymbol{S}^{-1}, \boldsymbol{V}, \mathbf{E}_{\boldsymbol{v}}, \mathbf{E}_{\boldsymbol{T}} \right\}, \iff \rangle \tag{53}$$

This relational system describes the characteristics of the rhythm of cyclic random processes. Taking into account the system of deterministic interdependencies presented in Tables 1 and 2, classes of rhythm functions (42) can be constructed using initially the known, admissible scale transformation functions $y^{-1}_{a_1,a_2,\ldots,a_K}(t)$.

By using the equation [22]:

$$y^{-1}_{a_1,a_2,\ldots,a_K}\left(t + T_{a_1,a_2,\ldots,a_K}(t,n)\right) - y^{-1}_{a_1,a_2,\ldots,a_K}(t) = 2\pi n \tag{56}$$

one can obtain an analytical expression for the parametric rhythm function $T_{a_1,a_2,\ldots,a_K}(t,n)$. The set of admissible parameters for the scale transformation function is a set of admissible parameters of the rhythm function.

Based on the above, it is possible to propose the algorithm which can be used for the construction of essential rhythm characteristics of cyclic random processes. It consists of following steps:

(1) Construction of parametric class $\boldsymbol{Y}$ of scale transformation functions $y^{-1}_{a_1,a_2,\ldots,a_K}(t)$ (a class of increasing functions). In the construction of this class, it is necessary to specify a set $\boldsymbol{A} \subseteq \boldsymbol{R}^K$ of admissible parameters $\left(a_k, k = \overline{1,K} \right) \in \boldsymbol{A}$.

(2) Construction of a parametric class $\boldsymbol{T}$ of rhythm functions $T_{a_1,a_2,\ldots,a_K}(t,n)$ on the basis of equality (56).

(3) Construction of a parametric class $\boldsymbol{S}$ of dynamic shift functions $s^{-1}_{a_1,a_2,\ldots,a_K}(t)$ obtained from relation.

$$s^{-1}_{a_1,a_2,\ldots,a_K}(t) = y^{-1}_{a_1,a_2,\ldots,a_K}(t) - t \tag{57}$$

(4) Construction of a parametric class $\boldsymbol{V}$ of instantaneous angular frequencies $v_{a_1,a_2,\ldots,a_K}(t)$ on the basis of equality

$$v_{a_1,a_2,\ldots,a_K}(t) = \frac{d}{dt} y^{-1}_{a_1,a_2,\ldots,a_K}(t) \tag{58}$$

(5) Construction of a parametric class $\mathbf{E}_{\boldsymbol{v}}$ of deviation functions of instantaneous angular frequencies $\varepsilon_{v\,a_1,a_2,\ldots,a_K}(t)$ on the basis of relation

$$\varepsilon_{v\,a_1,a_2,\ldots,a_K}(t) = v_{a_1,a_2,\ldots,a_K}(t) - 1 \tag{59}$$

(6) Construction of a parametric class $\mathbf{E}_{\boldsymbol{T}}$ of deviations of rhythm functions $\varepsilon_{T\,a_1,a_2,\ldots,a_K}(t,n)$ obtained from

$$\varepsilon_{T\,a_1,a_2,\ldots,a_K}(t,n) = T_{a_1,a_2,\ldots,a_K}(t,n) - 2\pi n \tag{60}$$

As can be observed from the dependencies (56)–(60), when the scale transformation function and the rhythm function of a cyclic random process are known, then all other characteristics of its rhythm can be easily calculated. The situation is more complicated with calculation of the rhythm function based on known scale transformation function according to formula (56).

5.2 Construction examples of interrelated parametric classes of rhythm characteristics of cyclic random processes

According to the approach described above, several parametric classes of rhythm functions will be considered.

5.2.1 System of parametric monomials of degree k

Applying described above algorithm, one-parametric classes $\boldsymbol{Y}_k^{-1}$ of scale transformation increasing functions $y_{a_1}^{-1}(t)$ are built for $t > 0$ and $a_1 \in (0,\infty) = \boldsymbol{A} \subset \boldsymbol{R}$, namely:

$$\boldsymbol{Y}_k^{-1} = \left\{ y_{a_1}^{-1}(t) = \frac{a_1 \cdot t^k}{k} : t > 0, a_1 \in \boldsymbol{A} \right\}, \text{where } k \in \boldsymbol{N}, \tag{61}$$

In fact, different one-parametric classes of scale transformation increasing functions are obtained for different integers k.

According to the second step, one-parametric classes $\boldsymbol{T}_k$ of rhythm functions $T_{a_1}(t,n)$ are calculated for $t > 0$ and $a_1 \in (0,\infty) = \boldsymbol{A} \subset \boldsymbol{R}$, namely, by substituting $y_{a_1}^{-1}(t) = \frac{a_1 \cdot t^k}{k}$ into the Eq. (56).

It leads to the following equation, relative to unknown rhythm function $T_{a_1}(t,n)$:

$$\frac{a_1}{k} \cdot \left[t + T_{a_1}(t,n)\right]^k - \frac{a_1 t^k}{k} = 2\pi n, \text{ where } n \geq 0, k \in \boldsymbol{N}.$$

After simple transformations, the following polynomial equation is obtained:

$$-t^k + \left[t + T_{a_1}(t,n)\right]^k - \frac{2\pi k n}{a_1} = 0$$

Thus.

$$\left[t + T_{a_1}(t,n)\right]^k = t^k + \frac{2\pi kn}{a_1}$$

The rhythm function is considered for two cases:

for $k = 2s + 1, s \in \boldsymbol{N}$:

$$T_{a_1}(t,n) = -t + \sqrt[k]{t^k + \frac{2\pi kn}{a_1}}$$

for $k = 2s, s \in \boldsymbol{N}$:

$$T_{a_1}(t,n) = -t \pm \sqrt[k]{t^k + \frac{2\pi kn}{a_1}}$$

Only functions for even values of k

$$T_{a_1}(t,n) = -t + \sqrt[k]{t^k + \frac{2\pi kn}{a_1}}$$

satisfy the rhythm function's assumptions (1)–(3).

So, one-parametric classes $\boldsymbol{T}_k$ of rhythm functions $T_{a_1}(t,n)$ can be written as:

$$\boldsymbol{T}_k = \left\{ T_{a_1}(t,n) = -t + \sqrt[k]{t^k + \frac{2\pi kn}{a_1}} : t > 0, a_1 \in \boldsymbol{A} \right\}, \text{where } k \in \boldsymbol{N} \tag{62}$$

According to the third step, classes $\boldsymbol{S}_k^{-1}$ of dynamic shift functions $s_{a_1}^{-1}(t)$ are built based on the Eq. (57), namely:

$$\boldsymbol{S}_k^{-1} = \left\{ s_{a_1}^{-1}(t) = \frac{a_1 \cdot t^k}{k} - t : t > 0, a_1 \in \boldsymbol{A} \right\}, \text{where } k \in \boldsymbol{N} \tag{63}$$

Then using the Eq. (58), one-parametric classes $\boldsymbol{V}_k$ of instantaneous angular frequencies $\nu_{a_1}(t)$ are constructed in the following way:

$$\boldsymbol{V}_k = \left\{ \nu_{a_1}(t) = a_1 \cdot t^{k-1} : t > 0, a_1 \in \boldsymbol{A} \right\}, \text{where } k \in \boldsymbol{N} \tag{64}$$

In the next step, sets $\mathbf{E}_{\boldsymbol{\nu} k}$ of deviation functions of instantaneous angular frequencies $\varepsilon_{\nu a_1}(t)$ are constructed on the bases of the Eq. (59):

$$\mathbf{E}_{\boldsymbol{\nu} k} = \left\{ \varepsilon_{\nu a_1}(t) = a_1 \cdot t^{k-1} - 1 : t > 0, a_1 \in \boldsymbol{A} \right\}, \text{where } k \in \boldsymbol{N} \tag{65}$$

According to the last step, one-parametric classes $\mathbf{E}_{\boldsymbol{T} k}$ of deviations of rhythm functions $\varepsilon_{T a_1}(t,n)$ are built based on the Eq. (60):

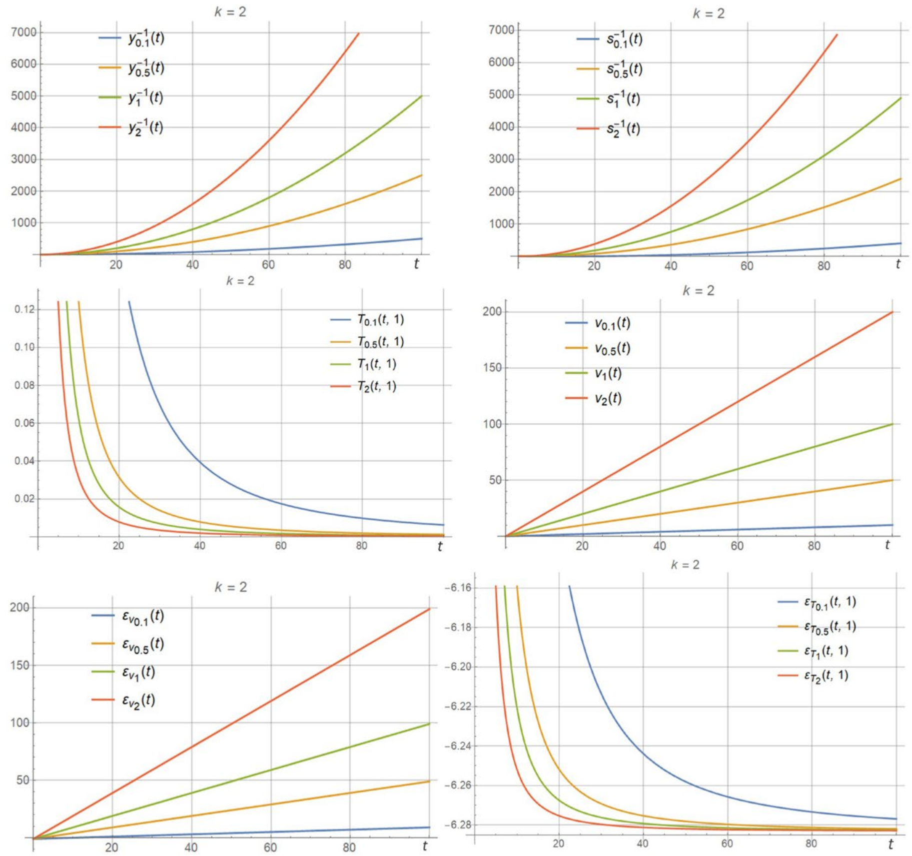

Fig. 2 Graphs of the characteristics $y_{a_1}^{-1}(t), s_{a_1}^{-1}(t), T_{a_1}(t,1), v_{a_1}(t), \varepsilon_{v a_1}(t), \varepsilon_{T a_1}(t,1)$ for $a_1 = 0.1, 0.5, 1, 2$ and $k{=}2$

$$\mathbf{E}_{Tk} = \left\{ \varepsilon_{T a_1}(t,n) = -t + \sqrt[k]{t^k + \frac{2\pi kn}{a_1}} - 2\pi n : t > 0, a_1 \in \boldsymbol{A} \right\}, \text{where } k \in \boldsymbol{N} \tag{66}$$

Let us consider an example of parametric characteristics of the rhythm function from the one-parameter classes of functions constructed above, i.e. $v_{a_1}(t) = a_1 \cdot t^{k-1}, t > 0$. Figure 2 show the graphs of characteristics $y_{a_1}^{-1}(t)$, $s_{a_1}^{-1}(t)$, $T_{a_1}(t,1)$, $v_{a_1}(t)$, $\varepsilon_{v a_1}(t)$, $\varepsilon_{T a_1}(t,1)$ of cyclic random process.

$\xi(\omega,t) = \xi_{2\pi}\left(\omega, \frac{a_1 \cdot t^k}{k}\right)$ ($\xi_{2\pi}(\omega,t) - 2\pi$-periodic random process), where $k{=}2$ is fixed and $a_1 = 0.1, 0.5, 1, 2$.

5.2.2 The system of parametric logarithmic functions

Let us consider the specific system of parametric logarithmic functions. According to the first step, certain three-parametric class $\boldsymbol{Y}^{-1}$ of scale transformation increasing functions $y_{a_1,a_2,a_3}^{-1}(t)$ is built:

$$\boldsymbol{Y}^{-1} = \left\{ y_{a_1,a_2,a_3}^{-1}(t) = a_1 \ln(a_2 t + a_3) : t > 0, (a_1,a_2,a_3) \in \boldsymbol{A} \right\} \tag{67}$$

where $\boldsymbol{A} = \boldsymbol{R}_+ \times \boldsymbol{R}_+ \times \boldsymbol{R}_+$.

According to the second step, three-parametric class $\boldsymbol{T}$ for $t > 0$ of rhythm functions $T_{a_1,a_2,a_3}(t,n)$ is constructed. Using the formula (56), the following sequence of transformations is obtained:

$$\int_t^{t+T(t,n)} \frac{a_1 a_2}{a_2 x + a_3} dx = a_1 \ln(a_2 x + a_3)\Big|_t^{t+T(t,n)}$$
$$= a_1 \ln(a_2(t + T(t,n)) + a_3) - a_1 \ln(a_2 t + a_3)$$
$$= 2\pi n,$$

where $n \geq 0$.

Simple computations lead to formula:

$$\ln \frac{a_2(t + T(t,n)) + a_3}{a_2 t + a_3} = \frac{2\pi n}{a_1}$$

Thus.

$$T(t,n) = T_{a_1,a_2,a_3}(t,n) = -t - \frac{a_3}{a_2} + \left(t + \frac{a_3}{a_2}\right) \cdot e^{2\pi n/a_1}$$

So, the three-parametric class $\boldsymbol{T}$ of rhythm functions $T_{a_1,a_2,a_3}(t,n)$ can be written as:

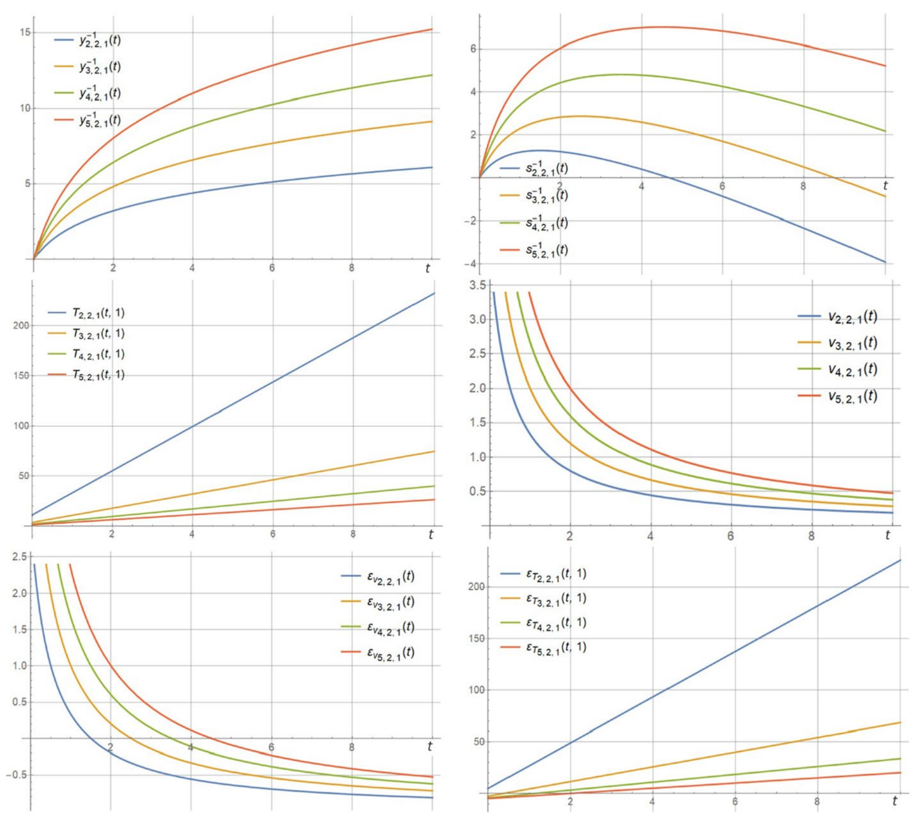

Fig. 3 Graphs of the characteristics $y^{-1}_{a_1,a_2,a_3}(t), s^{-1}_{a_1,a_2,a_3}(t), T_{a_1,a_2,a_3}(t,1), v_{a_1,a_2,a_3}(t), \varepsilon v_{a_1,a_2,a_3}(t), \varepsilon T_{a_1,a_2,a_3}(t,1)$ for fixed parameters $a_2 = 2, a_3 = 1$ and $a_1 = 2,3,4,5$

$$\boldsymbol{T}=\left\{T_{a_1,a_2,a_3}(t,n)=-t-\frac{a_3}{a_2}+\left(t+\frac{a_3}{a_2}\right)\cdot e^{\frac{2\pi n}{a_1}}:t>0,(a_1,a_2,a_3)\in\boldsymbol{A}\right\} \quad (68)$$

According to the third step, the set $\boldsymbol{S}^{-1}$ of dynamic shift functions $s^{-1}_{a_1,a_2,a_3}(t)$ is built based on Eq. (57), namely.

$$\boldsymbol{S}^{-1}=\left\{s^{-1}_{a_1,a_2,a_3}(t)=a_1\ln(a_2t+a_3)-t:t>0,(a_1,a_2,a_3)\in\boldsymbol{A}\right\} \quad (69)$$

Applying the Eq. (58), the class $\boldsymbol{V}$ of instantaneous angular frequencies $\nu_{a_1,a_2,a_3}(t)$ is constructed in the following way:

$$\boldsymbol{V}=\left\{\nu_{a_1,a_2,a_3}(t)=\frac{a_1a_2}{a_2t+a_3}:t>0,(a_1,a_2,a_3)\in\boldsymbol{A}\right\} \quad (70)$$

According to the fifth step, the class $\mathbf{E}_{\boldsymbol{\nu}}$ of deviation functions of instantaneous angular frequencies $\varepsilon_{\nu_{a_1,a_2,a_3}}(t)$ is built on the bases of the Eq. (59), namely:

$$\mathbf{E}_{\boldsymbol{\nu}}=\left\{\varepsilon_{\nu_{a_1,a_2,a_3}}(t)=\frac{a_1a_2}{a_2t+a_3}-1:t>0,(a_1,a_2,a_3)\in\boldsymbol{A}\right\} \quad (71)$$

The last step is the construction of the class $\mathbf{E}_{\boldsymbol{T}}$ for of deviations of rhythm functions $\varepsilon_{T_{a_1,a_2,a_3}}(t,n)$ on the bases of (60), namely:

$$\mathbf{E}_{\boldsymbol{T}}=\left\{\varepsilon_{T_{a_1,a_2,a_3}}(t,n)=-t-\frac{a_3}{a_2}+\left(t+\frac{a_3}{a_2}\right)\cdot e^{\frac{2\pi n}{a_1}}-2\pi n:t>0,(a_1,a_2,a_3)\in\boldsymbol{A}\right\} \quad (72)$$

Figure 3 shows the graphs of characteristics $y^{-1}_{a_1,a_2,a_3}(t)$, $s^{-1}_{a_1,a_2,a_3}(t)$, $T_{a_1,a_2,a_3}(t,1)$, $\nu_{a_1,a_2,a_3}(t)$, $\varepsilon_{\nu a_1,a_2,a_3}(t)$, $\varepsilon_{T a_1,a_2,a_3}(t,1)$ of cyclic random process $\xi(\omega,t)=\xi_{2\pi}(\omega,a_1\ln(a_2t+a_3))$, where parameters.

$a_2=2, a_3=1$ are fixed and $a_1=2,3,4,5$.

5.2.3 The system of parametric exponential functions

Let us consider the specific system of exponential functions. In the beginning, the three-parametric class $\boldsymbol{Y}^{-1}$ of scale transformation increasing functions $y^{-1}_{a_1,a_2,a_3}(t)$ is built, namely:

$$\boldsymbol{Y}^{-1}=\left\{y^{-1}_{a_1,a_2,a_3}(t)=a_1e^{a_2t+a_3}:t>0,(a_1,a_2,a_3)\in\boldsymbol{A}\right\} \quad (73)$$

where $\boldsymbol{A}=(0,\infty)\times(0,\infty)\times\boldsymbol{R}$.

According to the second step, class $\boldsymbol{T}$ of rhythm functions $T_{a_1,a_2,a_3}(t,n)$ is constructed. By using the formula (56), the following reasoning can be conducted:

$$\int\limits_t^{t+T(t,n)} a_1a_2e^{a_2x+a_3}dx=a_1e^{a_2x+a_3}\Big|\begin{matrix}t+T(t,n)\\ t\end{matrix}==a_1e^{a_2(t+T(t,n))+a_3}-a_1e^{a_2t+a_3}=2\pi n,$$

where $t>0, n\geq 0$.

After simple computations the following equation can be obtained:

$$e^{a_2 T(t,n)} = \frac{2\pi n}{a_1 e^{a_2 t + a_3}} + 1$$

Thus.

$$T(t,n) = T_{a_1,a_2,a_3}(t,n) = \frac{1}{a_2} \cdot \ln\left(\frac{2\pi n}{a_1 e^{a_2 t + a_3}} + 1\right)$$

So, the three-parametric class $\boldsymbol{T}$ of rhythm functions $T_{a_1,a_2,a_3}(t,n)$ can be written as:

$$\boldsymbol{T} = \left\{ T_{a_1,a_2,a_3}(t,n) = \frac{1}{a_2} \cdot \ln\left(\frac{2\pi n}{a_1 e^{a_2 t + a_3}} + 1\right) : t > 0, (a_1, a_2, a_3) \in \boldsymbol{A} \right\} \tag{74}$$

According to the third step, three-parametric class $\boldsymbol{S}^{-1}$ of dynamic shift functions $s^{-1}_{a_1,a_2,a_3}(t)$ is built based on Eq. (57), namely:

$$\boldsymbol{S}^{-1} = \left\{ s^{-1}_{a_1,a_2,a_3}(t) = a_1 e^{a_2 t + a_3} - t : t > 0, (a_1, a_2, a_3) \in \boldsymbol{A} \right\} \tag{75}$$

Based on Eq. (58), the class $\boldsymbol{V}$ of instantaneous angular frequencies $\nu_{a_1,a_2,a_3}(t)$ is constructed, namely:

$$\boldsymbol{V} = \left\{ \nu_{a_1,a_2,a_3}(t) = a_1 a_2 e^{a_2 t + a_3} : t > 0, (a_1, a_2, a_3) \in \boldsymbol{A} \right\} \tag{76}$$

The next step is the construction of the class $\mathbf{E}_{\boldsymbol{\nu}}$ of deviation functions of instantaneous angular frequencies $\varepsilon_{\nu_{a_1,a_2,a_3}}(t)$. Using the Eq. (59) one can obtain:

$$\mathbf{E}_{\boldsymbol{\nu}} = \left\{ \varepsilon_{\nu_{a_1,a_2,a_3}}(t) = a_1 a_2 e^{a_2 t + a_3} - 1 : t > 0, (a_1, a_2, a_3) \in \boldsymbol{A} \right\} \tag{77}$$

According to the last step, the class $\mathbf{E}_{\boldsymbol{T}}$ of deviations of rhythm functions $\varepsilon_{T_{a_1,a_2,a_3}}(t,n)$ is constructed on the basis of the Eq. (60), namely:

$$\mathbf{E}_{\boldsymbol{T}} = \left\{ \varepsilon_{T_{a_1,a_2,a_3}}(t,n) = \frac{1}{a_2} \cdot \ln\left(\frac{2\pi n}{a_1 e^{a_2 t + a_3}} + 1\right) - 2\pi n : t > 0, (a_1, a_2, a_3) \in \boldsymbol{A} \right\} \tag{78}$$

Graphs below show the characteristics $y^{-1}_{a_1,a_2,a_3}(t)$, $s^{-1}_{a_1,a_2,a_3}(t)$, $T_{a_1,a_2,a_3}(t,1)$, $\nu_{a_1,a_2,a_3}(t)$, $\varepsilon_{\nu a_1,a_2,a_3}(t)$, $\varepsilon_{T a_1,a_2,a_3}(t,1)$ of cyclic random process $\xi(\omega,t) = \xi_{2\pi}\left(\omega, a_1 e^{a_2 t + a_3}\right)$, where parameters $a_2 = 0.5, a_3 = 1$ are fixed and $a_1 = 0.1, 0.2, 0.3, 0.4$.

The proposed approach of construction interrelated parametric classes of rhythm characteristics of cyclic random processes is characterized by its universality, which makes it possible to generate another parametric classes needed for research. The system of parametric classes of the main characteristics of the rhythm function developed in this section significantly complements the known parabolic classes of scale transformation functions and rhythm functions, which was developed in [23], and expands the possibilities of applying the approximation approach to the estimation of the rhythm functions.

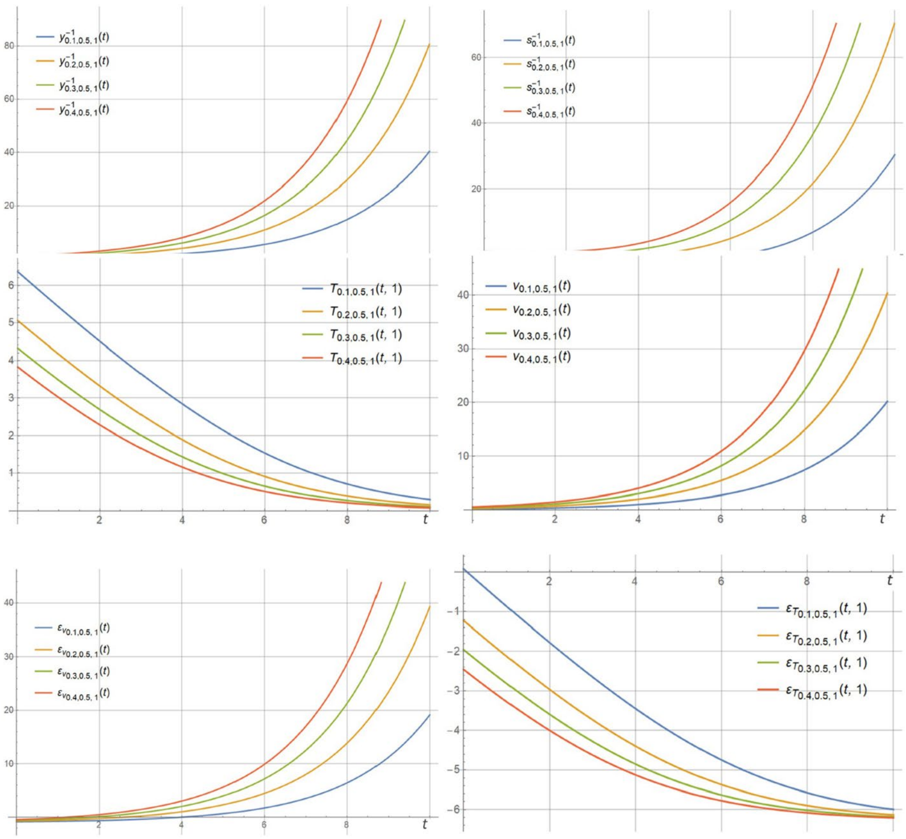

Fig. 4 Graphs of the characteristics $y^{-1}_{a_1,a_2,a_3}(t)$, $s^{-1}_{a_1,a_2,a_3}(t)$, $T_{a_1,a_2,a_3}(t,1)$, $v_{a_1,a_2,a_3}(t)$, $\varepsilon_{v\,a_1,a_2,a_3}(t)$,$\varepsilon_{T\,a_1,a_2,a_3}(t,1)$ for $a_2 = 0.5, a_3 = 1$and $a_1 = 0.1, 0.2, 0.3, 0.4$

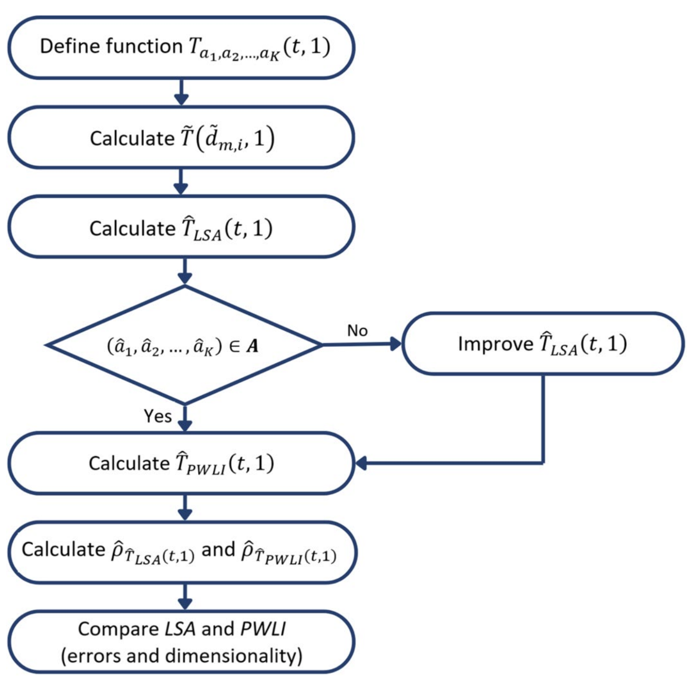

Fig. 5 Scheme of the computer simulation

6 Computer simulation experiments

6.1 The general scheme of the computer simulation

One of the important tools used in researching the main characteristics of the rhythm function estimation methods is the computer simulation experiment. In this paper, all computer simulation results were performed using MATHEMATICA software. Based on computer simulation experiment, the errors of approximation method and piecewise linear interpolation of the discrete rhythm functions are determined and compared, as well as the number of parameters of the approximant and interpolant. It is assumed in computer experiments, that the parametric class of possible rhythm functions of cyclic random process are known (predetermined). It is also assumed that the analytical rhythm function is given (all its parameters are known) and as well as the characteristics of the segmentation errors (Fig. 4).

The computer experiments itself are carried out according to the following steps (Fig. 5):

(1) Take the function $T_{a_1,a_2,\ldots,a_K}(t,1)$ belonging to some parametric class of functions $\boldsymbol{T}$ and satisfying the conditions (1)–(3) of the rhythm function of a cyclic random process (predefined rhythm function, i.e. $(a_1, a_2, \ldots, a_K) \in \boldsymbol{A}$.

(2) Computer simulation of the estimation $\widetilde{T}\left(\widetilde{d}_{m,i},1\right) = T_{a_1,a_2,\ldots,a_K}\left(\widetilde{d}_{m,i},1\right) + \varsigma\left(\widetilde{d}_{m,i}\right)$ of discrete rhythm function $T\left(\widetilde{d}_{m,i},1\right)$; $\varsigma\left(\widetilde{d}_{m,i}\right)$ is the discrete white noise with standard deviation $\sigma_{\varsigma}\left(\widetilde{d}_{m,i}\right)$ in two cases: $\sigma_{\varsigma}\left(\widetilde{d}_{m,i}\right) = \sigma_{\widetilde{T}\left(\widetilde{d}_{*},1\right)} = const$ or $\sigma_{\varsigma}\left(\widetilde{d}_{m,i}\right) = \sigma_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)}, i = \overline{1,N}$ (in the second case segmentation errors for zones are different).

(3) Calculation of the approximation $\widehat{T}_{LSA}(t,1) = \widehat{T}_{\widehat{a}_1,\widehat{a}_2,\ldots,\widehat{a}_K}(t,1)$ of the rhythm function

$T(t,1) = T_{a_1,a_2,\ldots,a_K}(t,1)$4. by approximating the estimation $\widetilde{T}\left(\widetilde{d}_{m,i},1\right)$ of the discrete rhythm function $T\left(\widetilde{d}_{m,i},1\right)$, using the method of least squares (by solving the optimization problem (45)).

(4) Checking whether the approximant satisfies the conditions (1)–(3) of the rhythm function by verifying whether $\left(\widehat{a}_k, k = \overline{1,K}\right) \in \boldsymbol{A}$. Namely, when a vector of parameters belongs to domain $\boldsymbol{A}$, then the estimation satisfies the conditions of the rhythm function. In the case when.

$\left(\widehat{a}_k, k = \overline{1,K}\right) \notin \boldsymbol{A}$6., it is necessary to estimate the level of deviation of the estimation from.

the conditions of the rhythm function, by calculating the first derivative $\frac{d\widehat{T}_{LSA}(t,1)}{dt}$ of approximants $\widehat{T}_{LSA}(t,1)$ and determining argument values in which the derivative of the approximant is no greater than minus one. Replacing these argument values with ones that would satisfy.

the appropriate condition for the rhythm function.

(5) Calculate estimation $\widehat{T}_{PWLI}(t,1)$ of rhythm function $T(t,1)$ by the piecewise linear interpolation method for the estimation $\widetilde{T}\left(\widetilde{d}_{m,i},1\right)$ of discrete rhythm function $T\left(\widetilde{d}_{m,i},1\right)$ according to work [19].

(6) Calculate the mean square error $\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$ of the estimation $\widehat{T}_{LSA}(t,1)$ according to the equation

$$\widehat{\rho}_{\widehat{T}_{LSA}(t,1)} = \frac{1}{K}\sum_{i=1}^{K}\left(T(t_i,1) - \widehat{T}_{LSA}(t_i,1)\right)^2, \tag{79}$$

where K is number of values of the rhythm function $T(t,1)$ and estimation $\widehat{T}_{LSA}(t,1)$, chosen as large as possible. The values $t_i = (i-1)\cdot\frac{\widetilde{d}_{M,1}}{K}$, $i = \overline{1,K}$.

(7) Calculate the mean square error $\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$ of the estimation $\widehat{T}_{PWLI}(t,1)$ according to the equation

$$\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)} = \frac{1}{K}\sum_{i=1}^{K}\left(T(t_i,1) - \widehat{T}_{PWLI}(t_i,1)\right)^2 \tag{80}$$

(8) Compare the errors $\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$ and $\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$.

(9) Compare the dimensionality of parameter vectors representing the rhythm function for interpolation and approximation approaches.

(10) Provide a conclusion about the accuracy of the approximation method based on the relation $\widehat{\rho}_{\widehat{T}_{LSA}(t,1)} \le \widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$ and additional conditions:

$$\widehat{\rho}_{\widehat{T}_{LSA}(t,1)} \le \sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)}, \forall i = \overline{1,N}, \tag{81}$$

$$\widehat{\rho}_{\widehat{T}_{LSA}(t,1)} \le \max_{i=\overline{1,N}} \sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)}, \tag{82}$$

$$\widehat{\rho}_{\widehat{T}_{LSA}(t,1)} \le \frac{1}{N}\cdot\sum_{i=1}^{N}\sigma^2_{\widetilde{T}\left(\widetilde{d}_{*,i},1\right)} \tag{83}$$

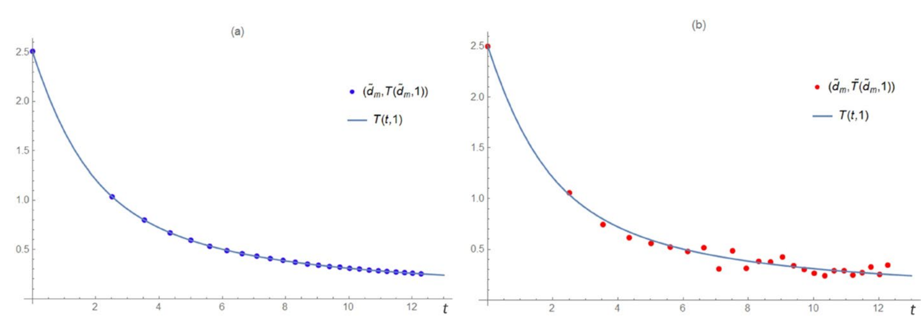

Fig. 6 Graphs of theoretical rhythm function $T(t,1)$, (a) discrete rhythm function $T\left(\widetilde{d}_m,1\right)$ and (b) realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m,1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m,1\right)$

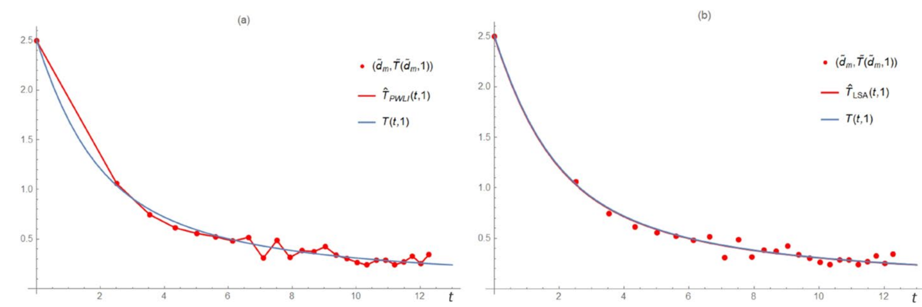

Fig. 7 Graphs of theoretical rhythm function $T(t,1)$ of cyclic random process, (a) interpolation $\widehat{T}_{PWLI}(t,1)$and (b) approximation $\widehat{T}_{LSA}(t,1)$ of discrete rhythm function $\widetilde{T}\left(\widetilde{d}_m,1\right)$

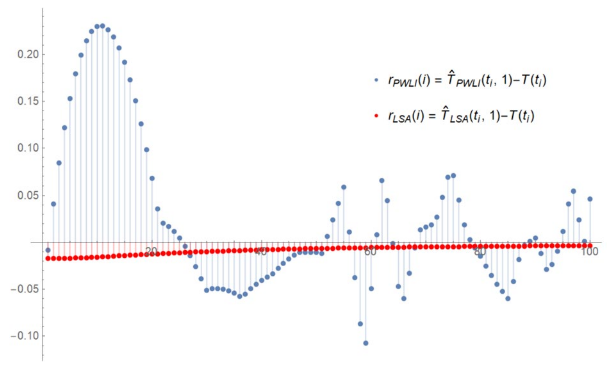

Fig. 8 Graph of residuals $r_{PWLI}(i) = \widehat{T}_{PWLI}(t_i,1) - T(t_i)$ and $r_{LSA}(i) = \widehat{T}_{LSA}(t_i,1) - T(t_i)$ for $i = \overline{1,100}$

6.2 Computer Experiment for the Estimation of the Rhythm Function from the Parametric Class of Monomials of degree 2

Let the rhythm function $T(t,1)$ of cyclic random process be known, namely, the rhythm function $T(t,1) = T_{a_1}(t,1)$ belongs to parametric class of monomials of degree 2. For parameter value $a_1 = 2$ such specified function is obtained

$$T(t,1) = T_2(t,1) = -t + \sqrt{t^2 + 2\pi}, \text{ where } t > 0 \tag{84}$$

Let the standard deviation of errors of automatic method segmentation of a cyclic random process be $\sigma_{\widetilde{d}_*} = \frac{0.05}{\sqrt{2}}$. According to formula (36), the standard deviation $\sigma_{\widetilde{T}\left(\widetilde{d}_*,n\right)} = 0.05$. Let us perform a computer simulation of the estimation $\widetilde{T}\left(\widetilde{d}_m,1\right)$ of the discrete rhythm function $T\left(\widetilde{d}_m,1\right) = T_2\left(\widetilde{d}_m,1\right)$, i.e.:

$$\widetilde{T}\left(\widetilde{d}_m,1\right) = T\left(\widetilde{d}_m,1\right) + \varsigma\left(\widetilde{d}_m\right) = -\widetilde{d}_m + \sqrt{\widetilde{d}_m^2 + 2\pi} + \varsigma\left(\widetilde{d}_m\right)$$

$$\widetilde{d}_1 = 0, \widetilde{d}_{m+1} = \widetilde{d}_m + \widetilde{T}\left(\widetilde{d}_m,1\right), m = \overline{1, M-1} \tag{85}$$

Table 3 Numerical results of considered approximation of $T(t,1)$, $\tilde{t}_i = (i-1)\cdot\Delta t$, $i = \overline{1,K}$, $\Delta t = \frac{\tilde{d}_{M,1}}{K}$, $K = 100$

Sym. No	Parameter table				$\hat{\rho}_{\hat{T}_{PWLI}(t,1)}$	$\hat{\rho}_{\hat{T}_{LSA}(t,1)}$
	$\hat{a}_1$	Standard error	*t*-statistic	*p*-value		
1	2.06051	0.054766	37.6234	$7.5\ 10^{-23}$	0.009171	0.000388
2	1.97607	0.044497	44.4093	$1.5\ 10^{-24}$	0.006611	0.000065
3	1.95881	0.029559	66.2687	$1.1\ 10^{-28}$	0.006496	0.000196
Avg of 500	2.00141	–	–	–	0.007141	0.000174

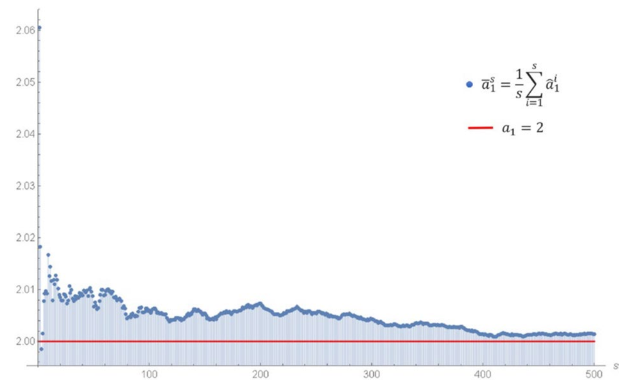

Fig. 9 Graph of averages $\overline{a}_1^s = \frac{1}{s}\sum_{i=1}^{s}\hat{a}_1^i$ for $s = \overline{1,500}$, where $a_1 = 2$ is assumed value of parameter

where $\varsigma\left(\tilde{d}_m\right)$ is the normally distributed discrete white noise with zero mean and standard deviation.

$$\sigma_\varsigma\left(\tilde{d}_m\right) = \sigma_{\tilde{T}\left(\tilde{d}_{*},n\right)} = \sqrt{2}\cdot\sigma_{\tilde{d}_*} = \sqrt{2}\cdot\frac{0.05}{\sqrt{2}} = 0.05 \tag{86}$$

The graphs of rhythm function $T(t,1)$, set $\left\{\left(\tilde{d}_m, T\left(\tilde{d}_m,1\right)\right), m = \overline{1,M-1}\right\}$ of points representing the discrete rhythm function $T\left(\tilde{d}_m,1\right)$ and set $\left\{\left(\tilde{d}_m, \tilde{T}\left(\tilde{d}_m,1\right)\right), m = \overline{1,M-1}\right\}$ of points representing the realization of the simulated estimation $\tilde{T}\left(\tilde{d}_m,1\right)$ of discrete rhythm function $T\left(\tilde{d}_m,1\right)$ of cyclic random process are presented in Fig. 6.

By approximating the simulated estimation $\tilde{T}\left(\tilde{d}_m,1\right)$ of the discrete rhythm function $T\left(\tilde{d}_m,1\right)$ using the method of least squares (by solving the optimization problem (45)) an estimation.

$\hat{T}_{LSA}(t,1) = \hat{T}_{\hat{a}_1}(t,1)$ of the rhythm function $T(t,1)$ was found. Graphs of estimation $\hat{T}_{LSA}(t,1)$ of rhythm function $T(t,1)$ and set $\left\{\left(\tilde{d}_m, \tilde{T}\left(\tilde{d}_m,1\right)\right), m = \overline{1,M-1}\right\}$ of points representing the realization of the simulated estimation $\tilde{T}\left(\tilde{d}_m,1\right)$ of discrete rhythm function $T\left(\tilde{d}_m,1\right)$ of cyclic random process are presented in Fig. 7b. The estimation $\hat{T}_{LSA}(t,1)$ of rhythm function $T(t,1)$ satisfies the conditions.

(1) – (3) of the rhythm function.

According to [19], the estimation $\widehat{T}_{PWLI}(t,1)$ of the rhythm function $T(t,1)$ was calculated by piecewise linear interpolation method for the simulated estimation $\widetilde{T}\left(\widetilde{d}_m,1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m,1\right)$. Graphs of estimation $\widehat{T}_{PWLI}(t,1)$ of rhythm function $T(t,1)$ and set $\left\{\left(\widetilde{d}_m,\widetilde{T}\left(\widetilde{d}_m,1\right)\right), m=\overline{1,M-1}\right\}$ of points representing the realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m,1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m,1\right)$ of cyclic random process are presented in Fig. 7a.

Errors of approximation and interpolation estimations of the rhythm function $T(t,1)$ are shown in Fig. 8.

As can be seen from Fig. 8, errors of approximation are smaller than errors of interpolation of the rhythm function $T(t,1)$. For the purpose of a more detailed analysis of errors of both approximation and interpolation of the rhythm function of a cyclic random process, selected accuracy characteristics are shown in Table 3 for exemplary three simulations and mean values for 500 simulations (Fig. 9).

As can be seen from Table 3, the mean square error of the interpolation $\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$ significantly exceeds the mean square error of approximation $\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$ (for example in the first simulation

$\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}=0.009171>\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}=0.000388$), which indicates a higher accuracy of the new method of estimation of the rhythm function. The estimates of the parameter $\widehat{a}_1$ satisfy the domain condition,i.e. $\widehat{a}_1>0$. Since the vector of parameters $(a_1,a_2,\ldots,a_K)$ of the approximant class is single-element, namely, determined by the parameter a_1, the developed method of approximation of the rhythm function has significant advantages over the known interpolation method, where the number of parameters is $2(M-1)$. Therefore, the approximation estimation of the rhythm function has significant advantages over the interpolation method.

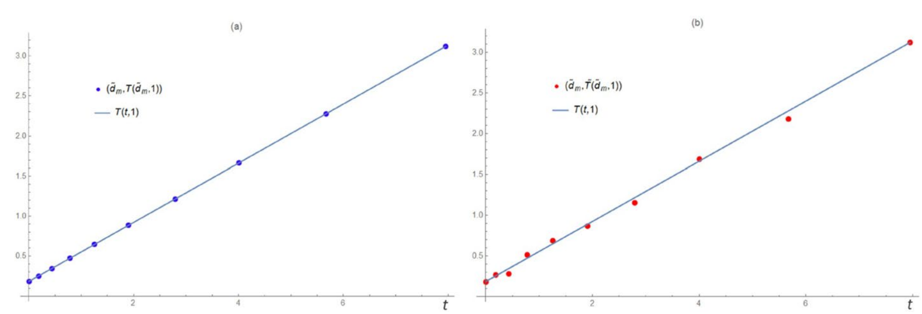

Fig. 10 Graphs of theoretical rhythm function $T(t,1)=T_{20,2,1}(t,1)$, (a) discrete rhythm function $T\left(\widetilde{d}_m,1\right)=T_{20,2,1}(\widetilde{d}_m,1)$ and (b) realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m,1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m,1\right)=T_{20,2,1}(\widetilde{d}_m,1)$

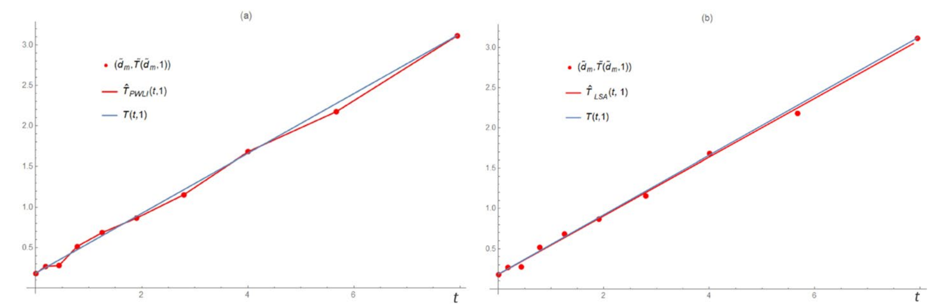

Fig. 11 Graphs of theoretical rhythm function $T(t, 1)$, (a) interpolation $\widehat{T}_{PWLI}(t, 1)$ and (b) approximation $\widehat{T}_{LSA}(t, 1)$ of discrete rhythm function $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of cyclic random process

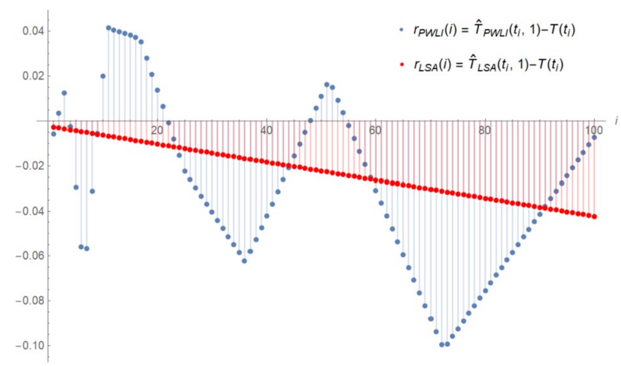

Fig. 12 Graphs of residuals $r_{PWLI}(i) = \widehat{T}_{PWLI}(t_i, 1) - T(t_i)$ and $r_{LSA}(i) = \widehat{T}_{LSA}(t_i, 1) - T(t_i)$ for $i = \overline{1, 100}$

Table 4 Numerical results of considered approximation of $T(t, 1)$, $\widetilde{t}_i = (i-1) \cdot \Delta t$, $i = \overline{1, K}$, $\Delta t = \frac{\widetilde{d}_{M,1}}{K}$, $K = 100$

Sym. No	Parameter table				$\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$	$\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$
	$\widehat{b}_0$	$\widehat{b}_1$	p_0-value	p_1-value		
1	0.181928	0.361989	6.5 10^{-5}	1.7 10^{-11}	0.002528	0.000923
2	0.225931	0.362101	7.5 10^{-6}	9.0 10^{-12}	0.002157	0.000449
3	0.198891	0.365749	1.7 10^{-4}	1.0 10^{-10}	0.001144	0.000061
Avg of 500	0.184486	0.368994			0.001775	0.000587

6.3 Computer experiment for the estimation of the rhythm function from the parametric class of logarithmic functions

Let the rhythm function $T(t, 1)$ of cyclic random process be known, namely, the rhythm function $T(t, 1)$ belongs to parametric class of logarithmic functions for $a_1 = 20, a_2 = 2, a_3 = 1$:

$$T(t,1) = T_{20,2,1}(t,1) = -t - \frac{1}{2} + \left(t + \frac{1}{2}\right)e^{\frac{2\pi}{20}}, t > 0 \tag{87}$$

It is assumed that the standard deviation $\sigma_{\widetilde{T}\left(\widetilde{d}_{*},n\right)} = 0.05$. Let us perform a computer simulation of the estimation $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of the discrete rhythm function $T\left(\widetilde{d}_m, 1\right) = T_{20,2,1}(\widetilde{d}_m, 1)$ according to (85).

The graphs of rhythm function $T(t,1) = T_{20,2,1}(t,1)$, set $\left\{\left(\widetilde{d}_m, T\left(\widetilde{d}_m, 1\right)\right), m = \overline{1, M-1}\right\}$ of points representing the discrete rhythm function $T\left(\widetilde{d}_m, 1\right)$ and set $\left\{\left(\widetilde{d}_m, \widetilde{T}\left(\widetilde{d}_m, 1\right)\right), m = \overline{1, M-1}\right\}$ of points representing the realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m, 1\right)$ of cyclic random process are presented in Fig. 10.

Approximation of the function

$$T(t,1) = -t - \frac{a_3}{a_2} + \left(t + \frac{a_3}{a_2}\right) \cdot e^{\frac{2\pi}{a_1}} = \frac{a_3}{a_2} \cdot \left(e^{\frac{2\pi}{a_1}} - 1\right) + \left(e^{\frac{2\pi}{a_1}} - 1\right) \cdot t$$

leads to simpler model.

$$\widehat{T}_{LSA}(t,1) = \widehat{T}_{b_0,b_1}(t,1) = b_0 + b_1 t$$

Where $b_0 = \frac{a_3}{a_2} \cdot \left(e^{\frac{2\pi}{a_1}} - 1\right)$ and $b_1 = e^{\frac{2\pi}{a_1}} - 1$ are the coefficients of the considered approximation model to find (the estimates $b_0 = 0.184554$ and $b_1 = 0.369108$ are expected). Function $\widehat{T}_{LSA}(t,1)$ must satisfy for $i = \overline{0, N-1}$ condition.

$$\frac{\widehat{T}_{LSA}\left(\widetilde{t}_{i+1}, 1\right) - \widehat{T}_{LSA}(\widetilde{t}_i, 1)}{\widetilde{t}_{i+1} - \widetilde{t}_i} > -1$$

Thus, in the considered case, the first derivative $\hat{T}_{LSA}{}'(t,1) = b_1$ satisfies $\hat{T}_{LSA}{}'(t,1) > -1$ for $b_1 > -1$.

Graphs of approximation $\widehat{T}_{LSA}(t,1)$ of rhythm function $T(t,1)$ and set $\left\{\left(\widetilde{d}_m, \widetilde{T}\left(\widetilde{d}_m, 1\right)\right), m = \overline{1, M-1}\right\}$ of points representing the realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m, 1\right)$ of cyclic random pro-

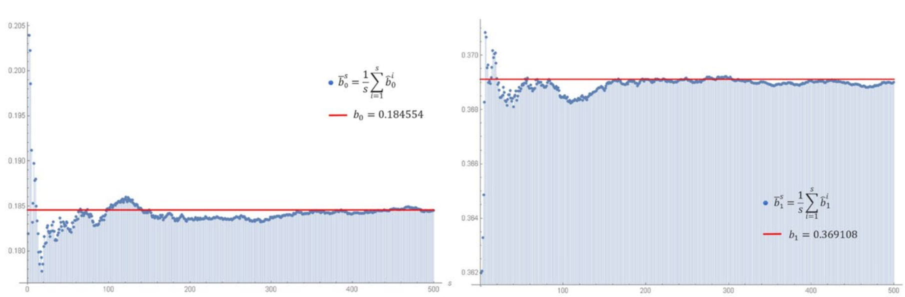

Fig. 13 Graphs of averages $\overline{b}_0^s = \frac{1}{s}\sum_{i=1}^{s}\widehat{b}_0^i$ and $\overline{b}_1^s = \frac{1}{s}\sum_{i=1}^{s}\widehat{b}_1^i$ for $s = \overline{1, 500}$, where $b_0 = 0.184554$and $b_1 = 0.369108$ are assumed values of parameters

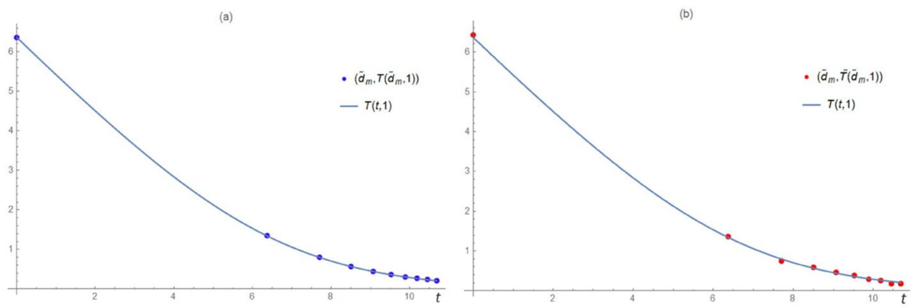

Fig. 14 Graphs of theoretical rhythm function $T(t,1) = T_{0.1,0.5,1}(t,1)$, (a) discrete rhythm function $T\left(\widetilde{d}_m, 1\right) = T_{0.1,0.5,1}(\widetilde{d}_m, 1)$ and (b) realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m, 1\right) = T_{0.1,0.5,1}(\widetilde{d}_m, 1)$

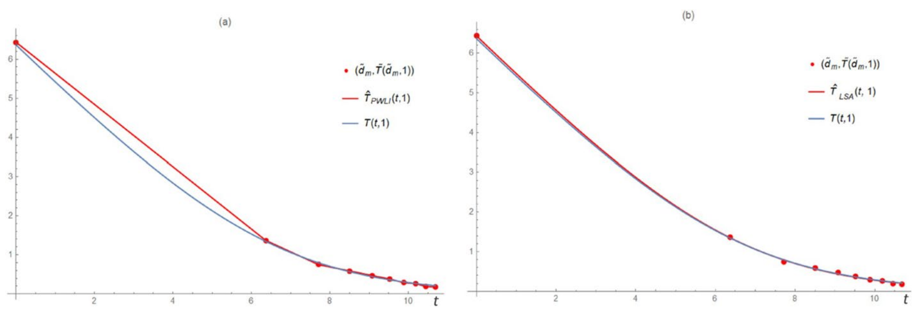

Fig. 15 Graphs of (a) interpolation $\widehat{T}_{PWLI}(t, 1)$ and (b) approximation $\widehat{T}_{LSA}(t, 1)$ of discrete rhythm function $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of cyclic random process

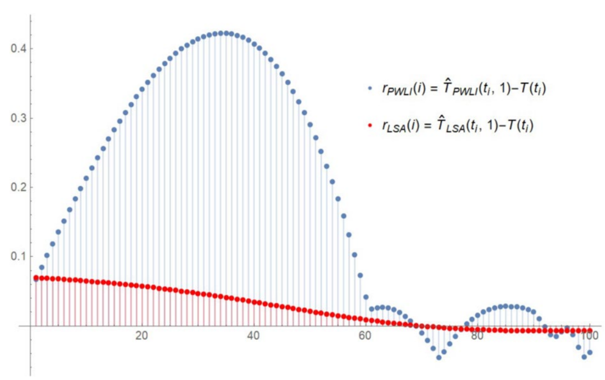

Fig. 16 Graphs of residuals $r_{PWLI}(i) = \widehat{T}_{PWLI}(t_i, 1) - T(t_i)$ and $r_{LSA}(i) = \widehat{T}_{LSA}(t_i, 1) - T(t_i)$ for $i = \overline{1, 100}$

Table 5 Numerical results of considered approximation of $T(t,1)$, $\widetilde{t}_i = (i-1)\cdot\Delta t$, $i = \overline{1, K}$, $\Delta t = \frac{\widetilde{d}_{M,1}}{K}$, $K = 100$

Sym. No	Parameter table				$\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$	$\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$
	$\hat{b}_0$	$\hat{b}_1$	p_0-value	p_1-value		
1	0.489381	21.5216	1.0×10^{-10}	8.8×10^{-10}	0.049879	0.000322
2	0.491108	22.0977	1.4×10^{-9}	1.1×10^{-8}	0.047723	0.000985
3	0.504312	23.1494	8.9×10^{-10}	6.8×10^{-9}	0.031868	0.001611
Avg of 500	0.499458	23.1346			0.046613	0.000921

cess are presented in Fig. 11b. Graphs of interpolation $\widehat{T}_{PWLI}(t,1)$ of rhythm function $T(t,1)$ and set $\left\{\left(\widetilde{d}_m, \widetilde{T}\left(\widetilde{d}_m,1\right)\right), m=\overline{1,M-1}\right\}$ of points representing the realization of the simulated estimation $\widetilde{T}\left(\widetilde{d}_m,1\right)$ of discrete rhythm function $T\left(\widetilde{d}_m,1\right)$ of cyclic random process are presented in Fig. 11a.

The visualizations of errors of approximation and interpolation of the rhythm function $T(t,1)$ are presented in Fig. 12.

As can be seen from Table 4, the mean square error of the interpolation $\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$ significantly exceeds the mean square error of approximation $\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$ (for example in the first simulation.

$\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)} = 0.002528 > \widehat{\rho}_{\widehat{T}_{LSA}(t,1)} = 0.000923$), which indicates a higher accuracy of the approximation method. Since the vector of parameters of the approximant class is two-element, namely, determined by the parameters b_0 and b_1, the developed method of approximation estimation the rhythm function of a cyclic random process has significant advantages over the known interpolation method, where the number of parameters is $2(M-1)$.

6.4 Computer experiment for the estimation of the rhythm function from the parametric class of exponential functions

Let the rhythm function $T(t,1)$ of cyclic random process be known, namely, the rhythm function $T(t,1) = T_{a_1,a_2,a_3}(t,1)$ from parametric class of exponential functions for $a_1 = 0.1, a_2 = 0.5, a_3 = 1$ is of the form:

$$T(t,1) = T_{0.1,0.5,1}(t,1) = 2\ln\left(1 + \frac{20\pi}{e^{0.5t+1}}\right).$$

Approximation of the function

$$T(t,1) = T_{a_1,a_2,a_3}(t,1) = \frac{1}{a_2}\cdot\ln\left(\frac{2\pi}{a_1 e^{a_2 t + a_3}} + 1\right) = \frac{1}{a_2}\cdot\ln\left(\frac{2\pi}{a_1 e^{a_3}} e^{-a_2 t} + 1\right).$$

leads to a simpler model

$$\widehat{T}_{LSA}(t,1) = \widehat{T}_{b_0,b_1}(t,1) = \frac{1}{b_0}\ln\left(b_1 e^{-b_0 t} + 1\right),$$

where $b_0 = a_2$ and $b_1 = \frac{2\pi}{a_1 e^{a_3}}$ are coefficients of considered approximation model to find (the expected estimates are $b_0 = 0.5$ and $b_1 = 23.1145$). Condition b_0,$b_1 > 0$ is met therefore the approximation is a valid rhythm function (Fig. 13).

Thus in the considered case the first derivative of approximation function

$$\hat{T}_{LSA'}(t,1) = \frac{-b_1 e^{-b_0 t}}{1 + b_1 e^{-b_0 t}}$$

satisfies the condition $\hat{T}_{LSA'}(t,1) > -1$ for $t, b_0, b_1 \in (0,\infty)$ (derivative is always negative, but greater than -1).

Figures 14, 15, 16 and 17 show the exemplary results of the computer simulation experiment related to the interpolation and approximation approaches to the estimation

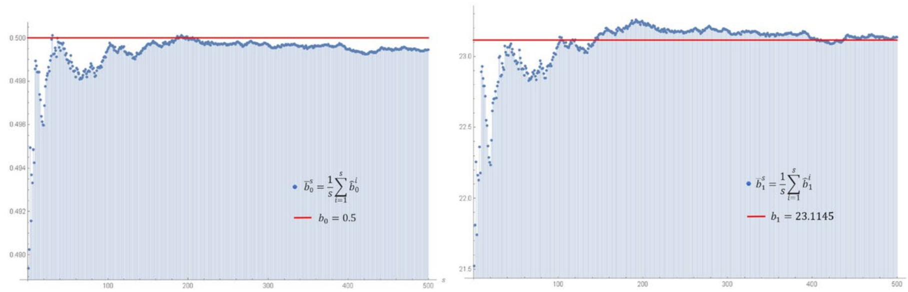

Fig. 17 Figures of averages $\overline{b}_0^s = \frac{1}{s}\sum_{i=1}^{s}\widehat{b}_0^i$ and $\overline{b}_1^s = \frac{1}{s}\sum_{i=1}^{s}\widehat{b}_1^i$ for $s = \overline{1,500}$, where $b_0 = 0.5$ and $b_1 = 23.1145$ are assumed values of parameters

of the rhythm function $T_{0.1,0.5,1}(t,1)$ on the basis of discrete rhythm function $\widetilde{T}\left(\widetilde{d}_m, 1\right)$ of cyclic random process.

As can be seen from Table 5, the mean square error of the interpolation $\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)}$ significantly exceeds the mean square error of approximation $\widehat{\rho}_{\widehat{T}_{LSA}(t,1)}$ (for example in the first simulation.

$\widehat{\rho}_{\widehat{T}_{PWLI}(t,1)} = 0.049879 > \widehat{\rho}_{\widehat{T}_{LSA}(t,1)} = 0.000322$), which indicates a higher accuracy of the new method of estimation of the rhythm function. Since the vector of parameters of the approximant class is two-element, namely, determined by the parameters b_0 and b_1, the developed method of approximation estimation the rhythm function of a cyclic random process has significant advantages over the known interpolation method, where the number of parameters is $2(M-1)$.

In summary, as can be seen from the above material, the results of all computer experiments indicate significant advantages of the approximation over the interpolation for the rhythm function of a cyclic random process. Such advantages take place both in the aspect of accuracy and in the aspect of the number of estimated parameters, which enables a significant reduction in the computational complexity of signal classification algorithms based on their rhythm parameters.

7 Conclusions

In the article an effective approximation method for estimating the rhythm function of a cyclic random process has been developed. In particular, the following scientific results were obtained.

The analytical dependencies between errors (absolute, RMS) of estimation of a discrete rhythm function and errors of segmentation of a cyclic random process into cycles and zones were constructed. This allowed the construction of confidence intervals for values of discrete rhythm function, if the segmentation results, namely, a set of estimated beginnings of cycles (or a set of estimated beginnings of zones in each cycle) and their corresponding errors are known. This made it possible to develop a procedure for calculating and controlling errors of rhythm function estimation for a cyclic random process as certain functions of errors of the segmentation method.

The general problem of approximate estimation of the rhythm function of a cyclic random process is formulated as a problem of parametric identification of a mathematical model of the rhythm function. It is a problem of selection of optimal parametric function (minimum of the mean square metric) from a predetermined class of functions that satisfy conditions of the rhythm function of a cyclic random process.

The universal approach to the construction of interrelated parametric classes of rhythm characteristics of cyclic random processes was developed. As a result of applying this approach, three different systems of interrelated parametric classes of rhythm characteristics of cyclic random processes were built, namely, the system of parametric monomials of degree k, the system of parametric logarithmic functions and the system of parametric exponential functions. New parametric classes of functions that satisfy the conditions of the rhythm function of a cyclic random process have been built. The formation of parametric classes of rhythm functions is significant not only from the point of view of building effective methods for the rhythm function's estimation, but also because of computer simulation of cyclic signals, methods of statistical evaluation of probabilistic characteristics, and is also important for the development of the theory of cyclic random processes.

On the basis of conducted series of computer experiments, significant advantages of proposed approximation approach to the evaluation of rhythm function of a cyclic process in comparison with interpolation methods have been established. The effectiveness of the new method is manifested in significant increase of accuracy of rhythm function estimation and reduction of the number of rhythm function estimation parameters in comparison with well-known interpolation methods. For example the computer experiment for the parametric class of monomials of degree 2 has shown the reduction of parameters from $2(M-1)$ in the case of piecewise linear interpolation to 1 for least square approximation, where M is the number of considered cycles. In this experiment, considering

500 simulations, average value of the mean square errors of the interpolation is equal to 0,007141 and significantly exceeds the corresponding value for approximation, which is 0.000174.

The results obtained in the work form a solid foundation for improvement of rhythm-adaptive methods, in particular in the area of statistical estimation and spectral analysis of probabilistic characteristics of cyclic random processes. In the perspective of further research, the effects of our work enable development of effective statistical methods for detecting hidden cyclic structures of investigated signals, which expands the problem of detecting hidden periodicities in the case of cyclic stochastic signals with an irregular rhythm.

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We were asked for the submission of PDF- file by Ms Maria Sabrina Greco. I pasted the e-mail we received below. Dear Dr Metelski, Your manuscript " Effective Least Squares Approximation Method for Estimating the Rhythm Function of Cyclic Random Process " has now been assessed. Regrettably, your manuscript has been rejected for publication in EURASIP Journal on Advances in Signal Processing because we do not accept doc file. Please submit a proper pdf file for review. Kind regards, Maria Sabrina Greco Editor in Chief EURASIP Journal on Advances in Signal Processing

Author contributions
Serhii Lupenko: Abstract, Introduction, Sections 2-5.1, 6.1, 7, ReferencesMałgorzata Wiatr: 5.2, 6.2-6.4, Figure 1, Figure 5Andrzej Metelski: 5.2, 6.2-6.4, Figure 1, Figure 5All authors reviewed the manuscript

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No datasets were generated or analysed during the current study.

Declarations

Competing interests
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References
1. J. A. Snoap, J. A. Latshaw, D. C. Popescu, and C. M. Spooner, Robust Classification of Digitally Modulated Signals Using Capsule Networks and Cyclic Cumulant Features, Proceedings of the Conference on Military Communications (MILCOM), Rockville, MD, November 2022.
2. J. A. Latshaw, D. C. Popescu, J. A. Snoap, and C. M. Spooner, Using Capsule Networks to Classify Digitally Modulated Signals with Raw I/Q Data, The 14th International Conference on Communications (COMM2022), Bucharest, Romania, June 2022.
3. J.A. Snoap, D.C. Popescu, J.A. Latshaw, C.M. Spooner, Deep-learning-based classification of digitally modulated signals using capsule networks and cyclic cumulants. Sensors **23**(12), 5735 (2023). https://doi.org/10.3390/s23125735
4. S.D. Reddy, R. Murugan, A. Nandi, Classification of arrhythmia disease through electrocardiogram signals using sampling vector random forest classifier. Multimed Tools Appl. **82**, 26797–26827 (2023). https://doi.org/10.1007/s11042-022-14304-x
5. A.R. Nematollahi, A.R. Soltani, Discrete time periodically correlated Markov processes. Probab. Math. Stat. **20**, 127–140 (2000)
6. T. Chen, L. Qiu, Linear periodically time-varying discrete-time systems: aliasing and LTI approximations. Syst. Control Lett. **30**, 225–235 (1997)
7. B.G. Marchenko, Linear Periodic Processes; Electrical Engineering; Pr. Institute of Electrodynamics, National Academy of Sciences of Ukraine: Kyiv, Ukraine, 1999; pp. 165–182. (in Ukrainian).
8. W.A. Gardner, The spectral correlation theory of cyclostationary time-series. Signal Process. **11**(1), 13–36 (1986). https://doi.org/10.1016/0165-1684(86)90092-7
9. A.Y. Dorogovtsev, Periodicity in distribution. I. Discrete systems. Int. J. Math. Math. Sci. **30**(2), 65–127 (2002). https://doi.org/10.1155/S0161171202011328
10. Dorogovtsev, A.Y. Stationary and periodic solutions of stochastic difference and differential equations in Banach space. In New Trends in Probability and Statistics; VSP: Utrecht, The Netherland, 1990; Volume 1, pp. 375–390.
11. W.A. Gardner, A. Napolitano, L. Paura, Cyclostationarity: half a century of research. Signal Process. **86**, 639–697 (2006). https://doi.org/10.1016/j.sigpro.2005.06.016
12. A. Napolitano, Cyclostationary processes and time series. Theory, applications, and generalizations. Elsevier, 2019. https://doi.org/10.1016/C2017-0-04240-4.
13. D. Abboud, M. Eltabach, J. Antoni, S. Sieg-Zieba, On the extraction of rolling-element bearing fault signature in speed-varying conditions, In: the eleventh international conference on condition monitoring and machinery failure prevention technologies, 2014.
14. A. Napolitano, Time-warped almost-cyclostationary signals: characterization and statistical function measurements. IEEE Trans. Signal Process. **65**, 5526–5541 (2017)
15. W.A. Gardner, Statistically inferred time warping: Extending the cyclostationarity paradigm from regular to irregular statistical cyclicity in scientific data. EURASIP J. Adv. Signal Process. 2018, 59.
16. S. Das, M.G. Genton, Cyclostationary processes with evolving periods and amplitudes. IEEE Trans. Signal Process. **69**, 1579–1590 (2021)
17. A. Napolitano, Modeling the electrocardiogram as oscillatory almost-cyclostationary process. IEEE Access **10**, 13193–13209 (2022). https://doi.org/10.1109/ACCESS.2022.3147500
18. Sun, R.-B.; Gao, Y.-S.; Yang, Z.-B.; Chen, X.-F. Restoring statistical cyclicity in rotating machinery vibrations using the oscillatory almost-cyclostationary model. Available at SSRN: https://doi.org/10.2139/ssrn.4438875.
19. S. Lupenko, R. Butsiy, N. Shakhovska, Advanced modeling and signal processing methods in brain-computer interfaces based on a vector of cyclic rhythmically connected random processes. Sensors. **23**, 760 (2023). https://doi.org/10.3390/s23020760
20. I. Lytvynenko, P. Maruschak, S. Lupenko, A. Menou, S. Panin, Software for segmentation, statistical analysis and modeling of surface ordered structures. AIP conference proceedings. mechanics, resource and diagnostics of materials and structures MRDMS-2016. 2016, 1785. https://doi.org/10.1063/1.4967033.
21. I. Lytvynenko, S. Lupenko, O. Nazarevych, G. Shymchuk, V. Hotovych, Mathematical model of gas consumption process in the form of cyclic random process. In Proceedings of the 16th IEEE international scientific and technical conference on computer sciences and information technologies (CSIT), Lviv, Ukraine, 22–25 September 2021; pp. 232–235.
22. S. Lupenko, The rhythm-adaptive Fourier series decompositions of cyclic numerical functions and one-dimensional probabilistic characteristics of cyclic random processes. Digit. Signal Proc. (2023). https://doi.org/10.1016/j.dsp.2023.104104
23. S. Lupenko, The mathematical model of cyclic signals in dynamic systems as a cyclically correlated random process. Mathematics. **10**, 3406 (2022). https://doi.org/10.3390/math10183406
24. N. Revanth, M. Anto Bennet, Detection of QRS Complexes in ECG Signal, 2023 IEEE international conference on integrated circuits and communication systems (ICICACS), Raichur, India, 2023, pp. 1–5, https://doi.org/10.1109/ICICACS57338.2023.10100203.

25. M.A. Belkadi, A. Daamouche, F. Melgani, A deep neural network approach to QRS detection using autoencoders. Expert Syst. Appl. **184**, 115528 (2021). https://doi.org/10.1016/j.eswa.2021.115528
26. A. Chen, Y. Zhang, M. Zhang, W. Liu, S. Chang, H. Wang, J. He, Q. Huang, A Real Time QRS Detection Algorithm Based on ET and PD Controlled Threshold Strategy. Sensors **20**, 4003 (2020). https://doi.org/10.3390/s20144003
27. M. Karri, C.S.R. Annavarapu, A real-time embedded system to detect QRS-complex and arrhythmia classification using LSTM through hybridized features. Expert Syst. Appl. **214**, 119221 (2023). https://doi.org/10.1016/j.eswa.2022.119221
28. S.M.I. Niroshana, S. Kuroda, K. Tanaka, W. Chen, Beat-wise segmentation of electrocardiogram using adaptive windowing and deep neural network. Sci. Rep. **13**, 11039 (2023). https://doi.org/10.1038/s41598-023-37773-y
29. Z. Wang, J. Wang, M. Chen, W. Yang, R. Fu, Deep regression network with sequential constraint for wearable ECG characteristic point location. IEEE Access **11**, 63487–63495 (2023). https://doi.org/10.1109/ACCESS.2023.3288700
30. I. Lytvynenko, S. Lupenko, P. Onyskiv, Method of Evaluation of Discrete Rhythm Structure of Cyclic Signals with the help of adaptive interpolation. In: proceedings of the 2020 IEEE 15th international scientific and technical conference on computer sciences and information technologies, CSIT 2020, Zbarazh, Ukraine, 23–26 September 2020; pp. 155–158.

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