Nonlinear pseudo-differential operators and stochastic equations on p-adic fields

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Historical notes

Starting point is the article of Kurt Hensel'1897 (1861–1941), introducing the notion of p-adic numbers (**part of algebraic number theory**).

Kronecker'1882 (in Berlin, supervisor of Hensel) published famous memoir on the foundations of this new branch of mathematics.

However, Hensel's idea was so novel and unexpected that it remains in the history of mathematics as a famous example of work developed in **almost complete isolation**.

Only fifteen years later the situation begin to change, with the introduction of simple **topological notions** in the field of p-adic numbers.

Let us consider the formal sum:

$$\sum_{i=0}^{\infty} a_i p^i, \quad \text{with } 0 \leq a_i \leq p-1, \;\; a_r \neq 0,$$

where the number p and the coefficients a_i are **natural integers**. Such a sum represents an **integer too**.

Hensel's idea was to include negative exponents, and this represents not only integers, but also rational numbers:

$$\sum_{i=r}^{\infty} a_i p^i, \quad \text{with } 0 \le a_i \le p-1,$$

where r may be negative integer. Moreover such a representation is unique!

$\mathbb{Q}_p$ – the set of all formal sums, and it is called the **field** of p-adic numbers. It contains $\mathbb{Q}$.

p-Adic numbers

Let p be a **prime number**. The field of p-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^{\nu}\frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and m, n are prime to p.

The absolute value $|x|_p$ takes the **discrete set** of non-zero values p^N, $N \in \mathbb{Z}$.

Note that by A.Ostrowski's theorem (1916) there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$.

Ultra-metric absolute value

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

1. $|x|_p = 0$ if and only if $x = 0$;
2. $|xy|_p = |x|_p \cdot |y|_p$;
3. $|x+y|_p \leq \max(|x|_p, |y|_p)$.

The latter property called the **ultra-metric inequality**. It implies that

$\mathbb{Q}_p$ is a **locally compact & totally disconnected** topological field in the metric $|x-y|_p$.

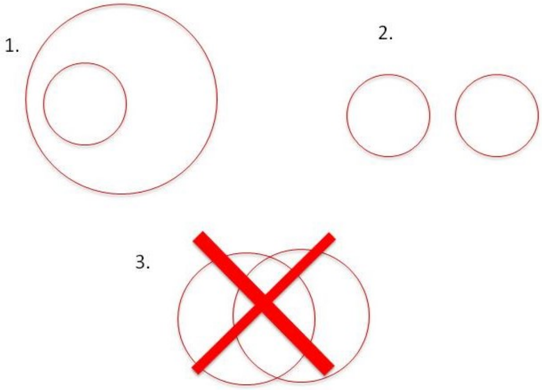
Note also the following consequence of the ultra-metric inequality:

$$|x+y|_p = \max(|x|_p, |y|_p), \quad \text{if} \quad |x|_p \neq |y|_p.$$

Some Unusual Properties of $\mathbb{Q}_p$

For some $a \in \mathbb{Z}_p$ the p-**adic ball** is

$$B_N(a) = \{x \in \mathbb{Z}_p : \|x - a\|_p \leq p^{-N}\}, \quad N \geq 0.$$



If $|x|_p = p^N$, then $x \in \mathbb{Q}_p$ admits a (unique) ***canonical representation***

$$x = p^{-N}\left(x_0 + x_1 p + x_2 p^2 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The series converges in the topology of $\mathbb{Q}_p$.

The ***fractional part*** of element $x \in \mathbb{Q}_p$ in canonical representation is given by:

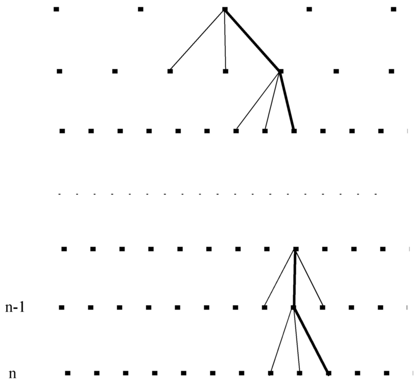
$$\{x\}_p = \begin{cases} 0, & \text{if} \quad N \leq 0 \quad \text{or} \quad x = 0; \\ p^{-N}\big(x_0 + x_1 p + \ldots + x_{N-1} p^{N-1}\big), & \text{if} \quad N > 0, \end{cases}$$

$\chi(x) = e^{2\pi i \{x\}_p}$ is an **additive character** of the field $\mathbb{Q}_p$.

Tree structure of p-adic numbers

A path, possibly infinite, on a rooted p-tree may be identified with a p-adic number $x \in \mathbb{Z}_p$ given by its canonical representation

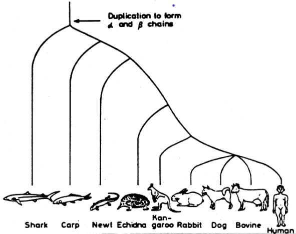
$$x = x_0 + x_1 p + x_2 p^2 + \cdots , \quad x_n \in \{0, 1, \ldots, p-1\}.$$



TAXÓNOMY

The classification can be represented as a dendrogram, or hierarchy, generally pictured as an inverted tree.

Going from the bottom up, several leaves (species) merge into a branch (genus); several such branches merge into a higher branch. **Higher taxa comprise a larger diversity of species than lower taxa.**



	Shark	Carp	Newt	Echidna	Kangaroo	Rabbit	Dog	Bovine	Human
Shark		85	84	84	80	75	80	75	79
Carp			74	75	71	71	67	65	68
Newt				71	67	69	65	64	62
Echidna					49	49	42	43	37
Kangaroo						37	33	26	27
Rabbit							28	25	25
Dog								28	23
Bovine									17
Human									

Amino Acid Differences

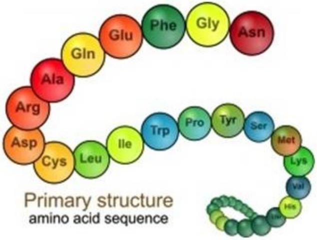
α chain

[1]The numbers of amino acid differences between the hemoglobin a chains of each pair of these animals

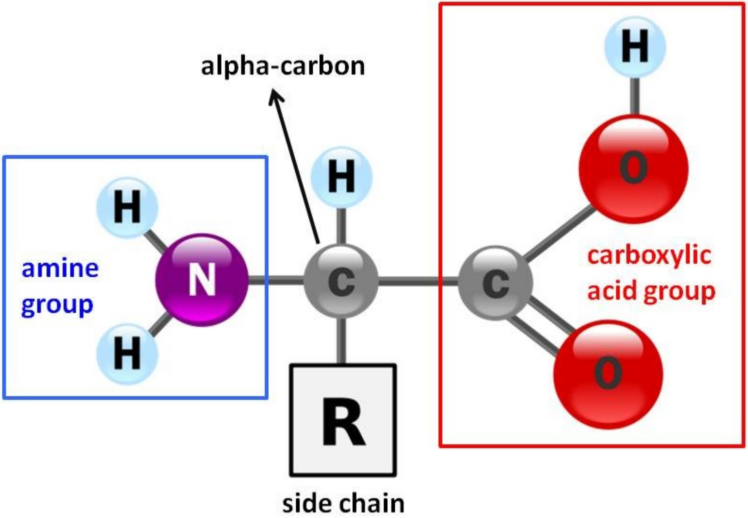
Protein folding

How proteins are made?

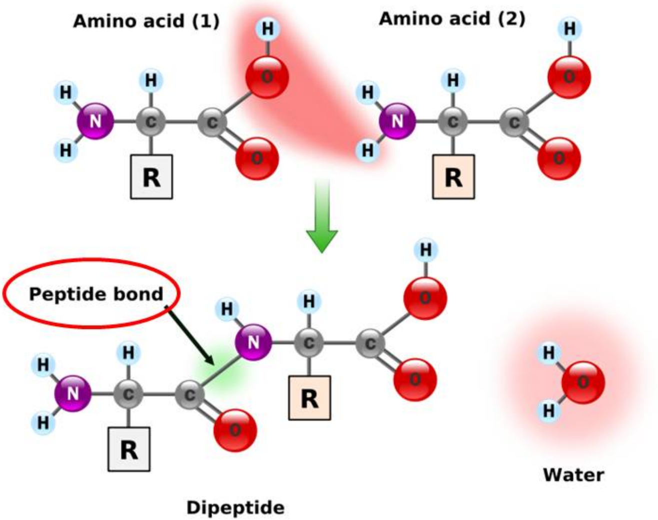
- polymer of repeated subunits: **amino acids** (амінокислоти) are the building blocks of proteins
- information about the sequence of amino acids for each specific protein is coded in the **DNA**
- **from 100 to several thousand** of amino acids in each protein



Amino acids


alpha-carbon
H
H
H
N
C
C
O
O
H
amine
group
carboxylic
acid group
R
side chain

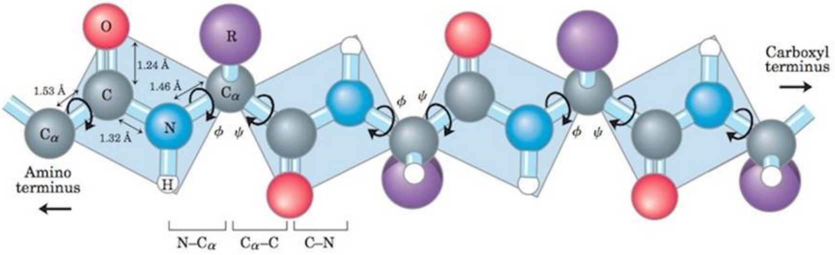
How amino acids are linked?



Primary Structure

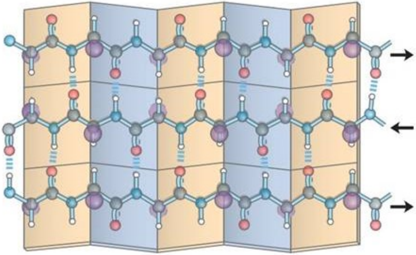
- Peptide bond is rigid and planar: C—N cannot rotate
- N—Cα and Cα—C bonds can rotate to define the dihedral angles *Φ* and *Ψ*, respectively

➡ **BACKBONE**= series of rigid plane, Cα as points of rotation

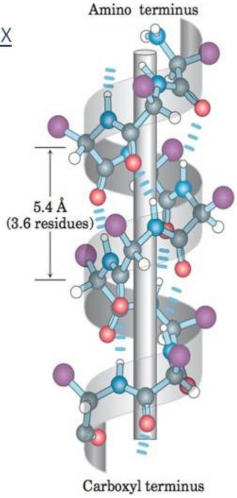


Secondary Structure

β SHEET

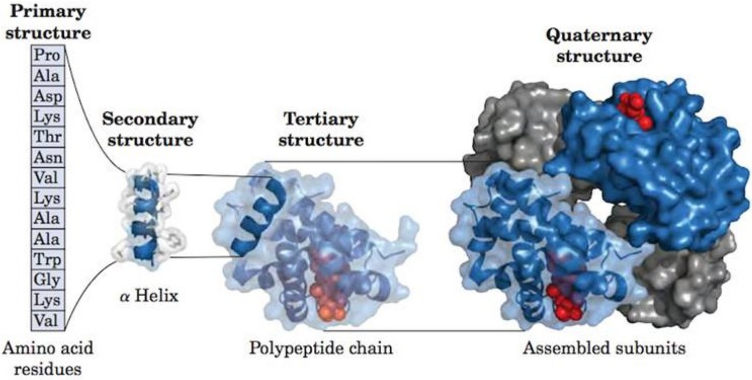


α HELIX



Quaternary Structure

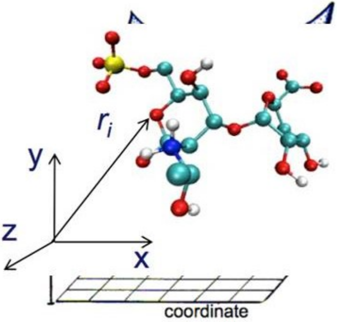
Quaternary structure results from interactions between several protein molecules of multisubunit proteins (eg. hemoglobin)



Molecular Mechanics

Potential Energy Function (PES) = multi-dimension energy function of molecular system coordinates

Objective: reach the **global minimum** of the PES, which is associated with the **native state** of a protein (various method and algorithms)



FORCE-FIELD

FORCE-FIELD = equations and parameters used to describe the Potential Energy Function

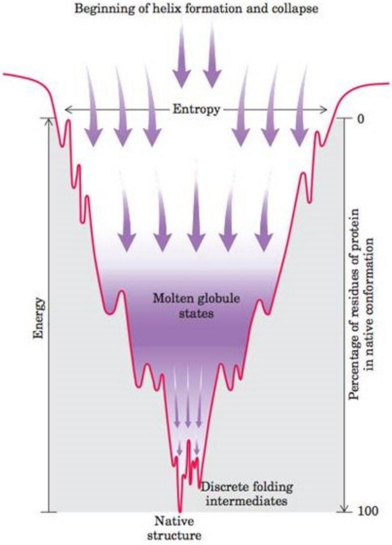
- each **atom** is simulated using a single particle, characterized by a charge and a radius
- **bonded interaction** (covalent bonds , angles, dihedrals)
- **non-bonded interactions** (van de Waals, electrostatic interaction, hydrogen bond)

$$V'_{nn}(R,R) = V(r_1, r_2, \ldots r_N) = \sum_{bonds} \frac{1}{2} k_l [l - l_0]^2 + \sum_{angles} \frac{1}{2} k_\theta [\theta - \theta_0]^2 +$$

$$+ \sum_{dihedrals} k_\varphi [1 + \cos(n\varphi - \delta)] + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \left[\frac{q_i q_j}{(4\pi\varepsilon_0 \varepsilon_r r_{ij})} + \frac{A(i,j)}{r_{ij}^{12}} - \frac{C(i,j)}{r_{ij}^{6}} \right] \ldots$$

Folding of Proteins

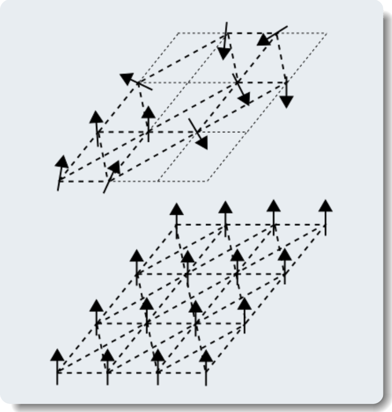
It corresponds to a thermodynamic path down to the **most favourable energy configuration** (decrease of entropy)



Spin Glass

In condensed matter physics, a spin glass is a **disordered mágnet**, where the magnetic spins of the component atoms (the orientation of the north and south magnetic poles in three-dimensional space) are not aligned in a regular pattern.

The term "glass" comes from an **análogy** between the **magnetic disorder** in a spin glass and the positional disorder of a conventional, chemical glass, e.g., a window glass.



Replication Technique

Parisi'1979 proposed to introduce the function

$$Z_n = \frac{1}{n} \int_{\Omega^N} Z^n(J) d\mu^{\otimes N}$$

and

$$\beta F = - \lim_{n \to 0} \left(Z_n - \frac{1}{n} \right)$$

Z_n is called the partition function of n identical replications. Parisi introduced as an order parameter the $n \times n$ matrix and an order parameter:

$$Q_i^{\alpha,\beta} = \langle \sigma_i^\alpha \sigma_i^\beta \rangle, \;\; \alpha \neq \beta$$

$$\bar{q} = \frac{1}{N} \sum_{i=1}^{N} \langle \langle \sigma_i \rangle^2 \rangle.$$

where the internal bracket indicates the thermodynamic expectation value at fixed J, while the external bracket indicates the mean value over J.

Parisi Matrix

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\alpha,\beta} Q^2_{\alpha\beta} < \infty;$$

$$\sum_{\beta=1}^{N} Q_{\alpha\beta} = \sum_{\beta=1}^{N} Q_{\gamma\beta}, \quad \alpha \neq \gamma;$$

$$-\lim_{n\to 0} \frac{1}{n} \sum_{\alpha,\beta} Q^2_{\alpha\beta} \geq 0.$$

$$\begin{pmatrix} 0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2 \\ q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2 \\ q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0 \\ q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0 \end{pmatrix}$$

Avetisov, Bikulov, Kozyrev'1999 The action of the replica matrix in the space of functions on $p^{-N}\mathbb{Z}/\mathbb{Z}$ takes the form:

$$Qf(x) = \int_{p^{-N}\mathbb{Z}/\mathbb{Z}} \rho(|x-y|_p) f(y)\, d\mu(y).$$

Avetisov, Bikulov, Kozyrev'1999

Master equation:

$$\frac{d}{dt}f(t,x) = \int\limits_{p^{-N}\mathbb{Z}/\mathbb{Z}} \big(f(t,y) - f(t,x)\big)\rho(|x-y|_p)\,d\mu(y)$$

For example, for $\rho(|x|_p) = |x|_p^{-1-\alpha}$ we have

$$\frac{d}{dt}f(t,x) = \int\limits_{p^{-N}\mathbb{Z}/\mathbb{Z}} \frac{f(t,y) - f(t,x)}{|x-y|_p^{1+\alpha}}\,d\mu(y)$$

Right-hand side is the Vladimirov operator.

ΨDE

The theory of linear partial pseudo-differential equations for complex-valued functions over non-Archimedean fields is well-established.

In contrary very little is known about **nonlinear** p-adic equations. In [A. Khrennikov and A. Kochubei, J. Fourier Anal. Appl.' 2018] it was considered a non-Archimedean analogue of the classical porous medium equation:

$$\frac{\partial u}{\partial t} + D^{\alpha}\big(\varphi(u)\big) = 0, \quad u = u(t,x), \quad t > 0, \quad x \in \mathbb{Q}_p, \tag{1}$$

where D^{α}, $\alpha > 0$ is the Vladimirov's fractional differentiation operator, i.e. ΨDO with symbol $|\xi|_p^{\alpha}$, or in terms of hypersingular integral representation:

$$\big(D^{\alpha}u\big)(x) = \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{u(x-y)-u(x)}{|y|_p^{\alpha+1}}\, dy, \quad u \in \mathcal{D}(\mathbb{Q}_p).$$

The Vladimirov-Taibleson operator

Denote by D^α the operator defined (initially) on the space $\mathcal{D}(\mathbb{Q}_p)$ by:

$$(D^\alpha \varphi)(x) = \mathcal{F}^{-1}_{\xi \to x} \left(|\xi|_p^\alpha \, \mathcal{F}_{x \to \xi} \, \varphi \right), \quad \alpha > 0.$$

Theorem 1. For $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ the pseudo-differential operator $D^\alpha \varphi$ is well-defined and has the following representation:

$$(D^\alpha \varphi)(x) = \frac{1}{\Gamma_p(-\alpha)} \int\limits_{\mathbb{Q}_p} \frac{\varphi(x-y) - \varphi(x)}{|y|_p^{\alpha+1}} \, dy,$$

where $\Gamma_p(\gamma)$ is the p-adic Gamma function:

$$\Gamma_p(\gamma) = \frac{1 - p^{\gamma-1}}{1 - p^{-\gamma}}.$$

Non-local Operator D^α in $L_2(\mathbb{Q}_p)$

The operator $\left(D^\alpha, Dom(D^\alpha)\right)$ is ***essential self-adjoint*** and positive and generates C_0-semigroup of contraction $T(t)$ in space $L_2(\mathbb{Q}_p)$:

$$T(t)u = \begin{cases} Z_t * u = \int_{\mathbb{Q}_p} Z(t, x-y)u(y)\,dy, & t > 0; \\ u, & t = 0. \end{cases}$$

Here $Z_t(x) = Z(t,x)$

$$Z(t,x) = \int_{\mathbb{Q}_p} e^{-t|\xi|_p^\alpha} \chi(-x\xi)\,d\xi, \quad \text{for } \ t > 0.$$

is the ***heat kernel*** or ***fundamental solution*** of the corresponding Cauchy problem:

$$\begin{cases} D_t u(t,x) + D^\alpha u(t,x) = f(t,x), & x \in \mathbb{Q}_p,\ t \in [0,T]; \\ u(0,x) = u_0, & x \in \mathbb{Q}_p. \end{cases} \qquad \textbf{(??)}$$

Connection with stochastic processes

The Markov process ξ_t on $\mathbb{Q}_p$

The function $Z(t,x)$ is the transition density of a time and space ***homogeneous Markov process*** ξ_t which is bounded, right-continuous and has no discontinuities ***other than jumps.*** Moreover the associated semigroup

$$\big(T(t)u\big)(x) = \int_{\mathbb{Q}_p} Z(t, x-\xi)\, u(\xi)\, d\xi$$

is ***Feller*** one. The transition probability of the process ξ_t is

$$P(t,x,B) = \begin{cases} \int\limits_B p\,(t,x,y)\, dy, & \text{for } t>0;\ x,y\in\mathbb{Q}_p,\ B\in\mathcal{E} \\ 1_B(x), & \text{for } t=0, \end{cases}$$

$p(t,x,y) := Z(t,x-y),$ for $t>0,\ x,y\in\mathbb{Q}_p,$
$\mathcal{E} = \big(\mathbb{Q}_p, |\cdot|_p\big)$ - complete non-Archimedean metric space.

The Markov process in the ball B_N

Let ξ_t be the Markov process on $\mathbb{Q}_p$ constructed above.

Suppose that $\xi_0 \in B_N$. Denote by $\xi_t^{(N)}$ the sum of all jumps of the process ξ_τ, $\tau \in [0, t]$, ***whose absolute values exceed*** p^N. Since ξ_t is right continuous process with left limits, $\xi_t^{(N)}$ is finite a.s. Moreover $\xi_0^{(N)} = 0$. Let us consider process

$$\eta_t = \xi_t - \xi_t^{(N)}.$$

Since the jumps of η_t never exceed p^N by absolute value, this process remain a.s. in the ball B_N.

Generator of the "process in the ball"

Theorem 4. If $\eta_t\big|_{t=0} = x$ and $\varphi \in \mathcal{D}(B_N)$, then

$$\frac{d}{dt}\mathbf{E}\,\varphi(\eta_t)\Big|_{t=0} = -\big(D_N^\alpha\varphi\big)(x) + \lambda_N\,\varphi(x),$$

$$\lambda_N = \int\limits_{|y|_p > p^N} \frac{dy}{|y|_p^{\alpha+1}}.$$

and operator D_N^α is defined by restricting D^α to the function φ supported in the ball B_N and the resulting function $D^\alpha\varphi$ is considered only on the ball B_N, i.e. $\big(D_N^\alpha\varphi\big)(x) = \big(D^\alpha\varphi\big)\restriction_{B_N},\;$ for $\;\varphi \in \mathcal{D}(B_N)$.

Remark. Theorem 4 actually states that the ***generator of the stochastic process*** η_t in the ball B_N equals:

$$\mathfrak{A}_\eta = D_N^\alpha - \lambda_N.$$

Nonlinear ΨDE in $\mathbb{Q}_p$

Crandall and Pierre'1982

Theory of the parabolic nonlinear equation [Crandall, Pierre'1982]

$$D_t u + A\big(\varphi(u)\big) = 0, \quad \Omega \subset \mathbb{R}^n \tag{2}$$

based on the theory of stationary equation

$$L\,u + \beta(x, u(x)) = f,$$

[Brézis, Strauss'1973].

Here L is unbounded linear operator in $L_1(\Omega)$, β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, Ω is σ-finite measurable space. Under some natural assumptions, the nonlinear operator $\mathcal{A}\varphi = A \circ \varphi$ is accretive and admits an ***m*-accretive extension** $\mathcal{A}_\varphi$, the generator of a contraction semigroup $\Rightarrow$ implies solvability of (2).

Nonlinear p-adic ΨDE in L_1

Considered a non-Archimedean analogue of the classical porous medium equation:

$$D_t u + D^\alpha\big(\varphi(u)\big) = 0, \quad u = u(t,x), \quad t > 0, \quad x \in \mathbb{Q}_p,$$

where function φ is a strictly monotone increasing continuous real function, $|\varphi(s)| \leq C|s|^m$ for $s \in \mathbb{R}$ ($C > 0$, $m \geq 1$).

$A = D^\alpha : Dom(A) \subset L_1(\mathbb{Q}_p) \to L_1(\mathbb{Q}_p)$ is linear, generating contraction semigroup. ***Nonlinear operator*** $\mathcal{A} = A\varphi$ is one with the domain:

$$Dom(\mathcal{A}) = \{u \in L_1(\mathbb{Q}_p) : \ \varphi(u) \in Dom(A)\}.$$

Let equation

$$D_t u + D^\alpha(\varphi(u)) = 0, \quad u = u(t,x), \quad t > 0, \ x \in \mathbb{Q}_p,$$

is considered as a nonlinear equation on $L_1(\mathbb{Q}_p)$

$$D_t u + A(\varphi(u)) = 0,$$

where the linear operator A is a generator of the contraction semigroup.

Theorem 6. [A. Khrennikov, A. Kochubei'2018]

The operator $\overline{A\varphi}$ is m-accretive, so that, for any initial function $u_0 \in L_1(\mathbb{Q}_p)$, the Cauchy problem for equation

$$D_t u + \overline{A\varphi}(u) = 0$$

has a unique **mild solution**. Here $\overline{A\varphi}$ is the closure of $A\varphi$.

Weak solution for the nonlinear equation in finite p-adic ball B_N

L_2-theory of nonlinear p-adic ΨDE

Considered a non-Archimedean analogue of the classical porous medium equation:

$$D_t u + D_N^{\alpha}\big(\varphi(u)\big) = 0, \quad u = u(t,x), \quad t > 0, \quad x \in B_N,$$

where function φ is a strictly monotone increasing continuous real function, $|\varphi(s)| \leq C|s|^m$ for $s \in \mathbb{R}$ ($C > 0$, $m \geq 1$).

$A = D^{\alpha} : Dom(A) \subset L_2(B_N) \to L_2(B_N)$ is linear, generating contraction semigroup. ***Nonlinear operator*** $\mathcal{A} = A\varphi$ is one with the domain:

$$Dom(\mathcal{A}) = \{u \in L_2(B_N) : \ \varphi(u) \in Dom(A)\}.$$

Definition of weak solution

Definition 6. The function $u \in C([0,T], H_{-1})$ is an H_{-1} - solution to equation

$$\begin{cases} D_t u + D_N^{\alpha}\big(\varphi(u)\big) = 0, \quad t \in [0,T]; \\ u(0) = u_0, \end{cases}$$

if $\varphi(u) \in L_1([0,T], H_1)$ and

$$\int_0^T \int_{B_N} u\, D_t\psi \, dy\, dt = \int_0^T \int_{B_N} \varphi(u)\, D_N^{\alpha}\psi \, dy\, dt$$

for any $\psi \in \mathcal{D}(B_N)$, where $\mathcal{D}(B_N)$ is the space of locally constant functions with compact support in B_N.

Main result

Theorem 7. [O.A., A. Kochubei, O.Nikitchenko'2023]
For any $u_0 \in H_{-1}$ there exists a unique H_{-1}-solution $u \in C([0,T], H_{-1})$ of the nonlinear Cauchy problem for porous medium equation in p-adic ball B_N for every $T > 0$. Moreover we have

$$t\varphi(u) \in L_\infty([0,T], H_{-1})$$

$$t\, D_t u \in L_\infty([0,T], H_{-1})$$

We also have that $u\varphi(u) \in L_1([0,T] \times B_N)$ and the solution map $S_t : u_0 \to u(t)$ defines a semigroup of non-strict contractions in H_{-1}

$$\|u(t) - v(t)\|_{H^{-1}} \leq \|u(0) - v(0)\|_{H^{-1}},$$

which turns out to be also compact in H_{-1}.

Thank you very much for your attention!

Description of the space H_{-1}

The dual space H_{-1} to H_1 is obtained as the closure of $L_2(B_N)$ w.r.t. the scalar product

$$(f, g)_{-1} = (f, g)_{L_2(B_N)} + ([D_N^\alpha]^{-1} f, g)_{L_2(B_N)}.$$

But this is not sufficient information!

The analog of Aronszain-Gagliardo-Slobodecki spaces in p-adics

Definition 7. Let $u \in L_2(B_N)$ and $s \in (0, 1)$. We say that function u belongs to the **Aronszain-Gagliardo-Slobodecki** space $H^s_{AGS}(B_N)$ if the following norm is finite:

$$\|u\|_{H^s_{AGS}} = \|u\|_{L_2(B_N)} + [u]_s,$$

where

$$[u]_s^2 = \int_{B_N} \int_{B_N} \frac{|u(x) - u(y)|^2}{|x - y|_p^{2s+1}} \, dx \, dy.$$

Recall:

$$(D^\alpha \varphi)(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{\varphi(x - y) - \varphi(x)}{|y|_p^{\alpha+1}} \, dy,$$

Pontryagin duality and p-adics Sobolev spaces

First remark that B_N is a **compact subgroup** of $\mathbb{Q}_p$ and its annihilator $\{\xi \in \mathbb{Q}_p : \chi(\xi x) = 1 \text{ for all } x \in B_N\}$ coincides with the ball B_{-N}. By the **Pontryagin duality theorem**, the dual group $\widehat{B}_N$ to B_N is isomorphic to the discrete group $\mathbb{Q}_p/B_{-N}$ consisting of cosets

$$p^m(r_0 + r_1 p + \cdots + r_{N-m-1}p^{N-m-1}) + B_{-N}, \quad r_j \in \{0, 1, \ldots, p-1\},$$

$m \in \mathbb{Z},\ m < N.$

This isomorphism means that any nontrivial continuous character of B_N, which has the form $\chi(\xi \cdot x)$, $x \in B_N$, where $|\xi|_p > p^{-N}$ and $\xi \in \mathbb{Q}_p$, is considered as a representative of the class $\xi + B_{-N}$. Moreover the value $|\xi|_p$ is the same for any representative of the class.

Fourier transform on the compact subgroup

The normalized Fourier transform on the compact subgroup B_N given by the formula:

$$\widehat{f} \equiv (\mathcal{F}_N f)(\xi) = p^{-N} \int\limits_{B_N} \chi(x\xi) f(x)\, dx, \quad \xi \in (\mathbb{Q}_p/B_{-N}) \cup \{0\}.$$

It follows that $\mathcal{F}_N f$ can be understood as a function on $\mathbb{Q}_p/B_{-N}$. Since $\mathcal{F} : \mathcal{D}(\mathbb{Q}_p) \to \mathcal{D}(\mathbb{Q}_p)$, therefore the Fourier transform $\mathcal{F}$ maps $\mathcal{D}(B_N)$ onto the set of functions on the discrete set $\widehat{B_N}$ with only a finite number of nonzero values.

Sobolev-Pontryagin spaces

Definition 8. The Sobolev-Pontryagin space $H^\alpha(B_N)$ consists of such functions $f \in L_2(B_N)$ that

$$\|f\|^2_{H^\alpha(B_N)} = \int\limits_{\widehat{B}_N} |\widehat{f}(\xi)|^2 \left(1 + |\xi|_p^2\right)^\alpha d\xi < \infty$$

where $\widehat{f} = \mathcal{F}_N f$ is the Fourier transform in the p-adic ball B_N.

The theorem on isomorphism

Theorem 8. [O.A., A. Kochubei, O. Nikitchenko' 2023]
Let $\alpha \in (0, 1)$. Then

$$\|u\|^2_{H^\alpha(B_N)} \asymp \|u\|^2_{H^\alpha_{AGS}} \asymp H_1.$$