

Article

Abstract Cyclic Functional Relation and Taxonomies of Cyclic Signals Mathematical Models: Construction, Definitions and Properties

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Abstract: This work is devoted to the procedure of the construction of an abstract cyclic functional relation, which summarizes and extends the known results for a cyclically correlated random process and a cyclic (cyclically distributed) random process to the case of arbitrary cyclic functional relations. Two alternative definitions of the abstract cyclic functional relation are given, and the fundamental properties of its cyclic and phase structures are presented. The theorem on the invariance of cyclicity attributes of an abstract cyclic functional relation to shifts of its argument, and which are determined by the rhythm function of this functional relation, is formulated and proved. This theorem gives the sufficient and necessary conditions that the rhythm function of an abstract cyclic functional relation must satisfy. By specifying the range of values and attributes of the cyclicity of an abstract cyclic functional relation, the definitions of important classes of cyclic functional relations are formulated. A deductive approach to building a wide system of taxonomies of classes of deterministic, stochastic, fuzzy and interval cyclic functional relations as potential mathematical models of cyclic signals is demonstrated. A comparative analysis of an abstract cyclic functional relation with the known mathematical models of cyclic signals was carried out. The results obtained in the article significantly expand and systematize the mathematical tools of the description of cyclic signals and are the basis for the development of effective model-based technologies for processing and computer simulation of signals with a cyclic space-time structure.



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MSC: 37H05; 60G12

1. Introduction

It is well known that the intensive development of the digital signal processing industry has penetrated into all spheres of human existence, automating the previously routine, resource-consuming and long-term procedures of information processing, ensuring a high level of accuracy, validity and reliability of such procedures in telecommunications, energy, transport, medicine, the economy and other areas. This state of affairs also occurs in the problems of processing (estimation of characteristics, detection, classification, clustering and forecasting) the signals (processes and phenomena) of space-time structure, in particular, cardiac signals (electrocardiosignals, magnetocardiosignals, phonocardiosignals, photoplethysmocardiosignals, etc.), vibration signals of mechanisms and machines in rotation modes, astrophysical radiation signals of space objects (i.e., stars, quasars), modulated and demodulated telecommunication signals, gas and electricity consumption processes and cyclical economic processes.

The modern technologies (methods, algorithms, hardware and software) of digital processing of signals with a cyclic space-time structure can be conditionally divided into three large groups, namely, model-based, data-based and hybrid digital technologies. Model-based technologies are clearly oriented to previously developed mathematical models of the investigated signals; data-based technologies mostly do not require the prior development of a mathematical model of a cyclic signal, since they use machine learning methods based on a large amount of input data. Hybrid technologies, in a certain way, integrate processing procedures within the framework of model-based and data-based approaches. Despite the fact that data-based methods and algorithms for the processing of cyclic signals do not explicitly require the construction of a mathematical model of cyclic signals, they require the preliminary development of a mathematical metamodel of the process of automated generation of an optimal or quasi-optimal data-based algorithm for processing cyclic signals. This metamodel should clearly reflect the architecture of the information and computing environment of the data-based technology and the procedure for evaluating the optimal (quasi-optimal) parameters of the signal processing algorithm. In order to ensure the efficiency (accuracy, reliability and low time computational complexity) of processing cyclic signals, it is desirable that the architecture of the information and computing environment of hybrid technology also has formal means of taking into account the regularities of the space-time structure of the investigated signals, which also requires the development of appropriate mathematical model cyclic signals.

In this article, we will focus directly on the model-based approach to cyclic signal processing technologies. Therefore, according to the work [1], we will present the main components and stages of the development of information systems for processing cyclic signals using a model-based approach as shown in Figure 1.

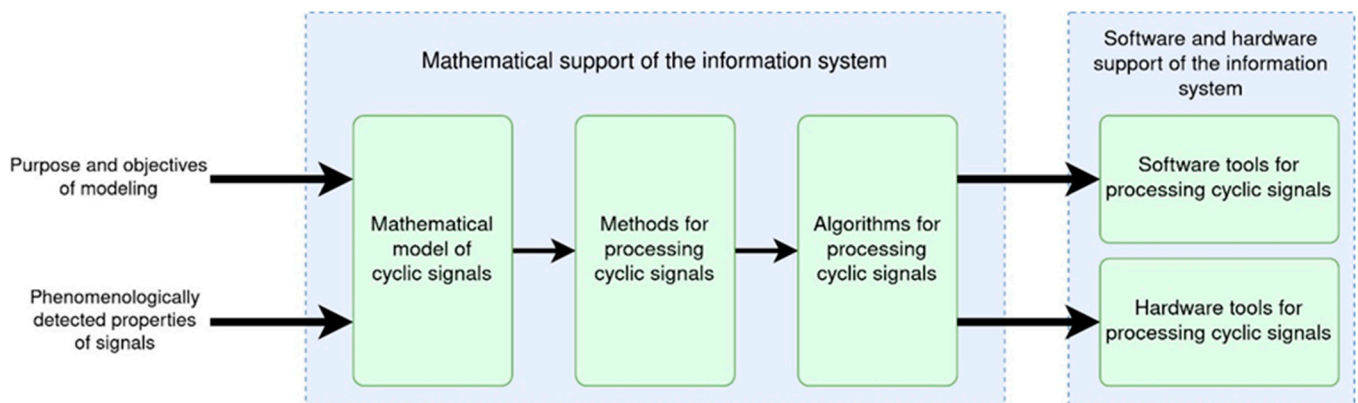


Figure 1. Stages of development of an information system for digital processing of cyclic signals using a model-based approach [1].

As can be seen from this figure, the initial and main stage of designing information systems for processing and simulating cyclic signals is the creation of their mathematical models that adequately reflect the important, from the point of view of research tasks, aspects of their space-time structure. In the framework of the model-based approach, the mathematical model largely determines the potential and effectiveness of the information technologies being created, and to some extent, determines the structure of the software and hardware components of the designed information system. The quality of the mathematical model of cyclic signals (see Figure 2) significantly determines the accuracy and reliability of the methods of their processing in the information system and the level of informativeness of the diagnostic and authentication methods, as well as the reliability of the decisions made.

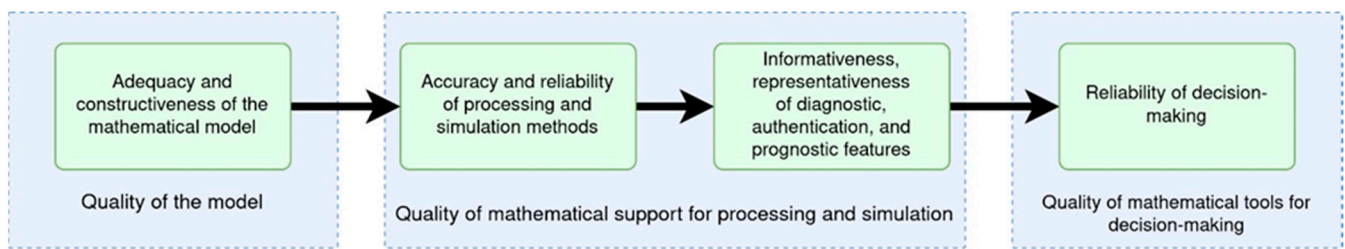


Figure 2. Causal relationships regarding the quality of the information system for processing cyclic signals [1].

Today, there are a large number of mathematical models of cyclic signals, which are used as a logical–formal core of model-based technologies for digital processing of these signals. Mathematical models of cyclic signals can be conditionally divided into two large and interrelated types: constructive mathematical models that clearly reflect the regularities and mechanisms of the construction (i.e, formation, generation) of cyclic signals, and structural mathematical models that clearly reflect the regularities and organization of the structure (space and/or time and/or frequency structures) of the studied cyclic signals. Constructive mathematical models of cyclic signals are mainly presented as differential and integral equations for continuous parameter signals (linear and nonlinear, deterministic and stochastic differential equations and linear periodic random processes and fields) [2–4] or as difference equations, autoregression and moving-average models for cyclic signals of a discrete parameter [5,6]. In the article, we will focus only on structural mathematical models of cyclic signals, since often, in the practical tasks of developing digital technologies, the mechanisms of the generation of these signals may not be taken into account. However, the structure of the cyclic signal itself and its parameters are mainly important objects of the structural–parametric identification procedures of these signals, since they are the carriers of information necessary to solve the corresponding problems of signal processing.

Tables 1–4 present the main (not all of them) structural mathematical models of cyclic signals and some of their properties. The theory of mathematical modeling of cyclic signals received the greatest development through the use of the idea of periodicity, namely, periodic deterministic and stochastic functions. Table 1 presents the main (not all of them) structural mathematical models based on a periodic pattern.

Table 1. Structural mathematical models of cyclic signals based on a periodic pattern.

	Properties of Cyclic Signals						
	One-Dimensional Cyclicity of Signals	Multidimensional Cyclicity of Signals	Two or More Cyclicity Attributes	Uncertainty, Inaccuracy, Randomness of Signals	Variability of the Rhythm of Signals	Stochasticity of the Rhythm Variability	Rhythmic Connection between Several Cyclic Signals
Harmonic function [7]	+	-	-	-	-	-	-
Periodic deterministic function [8]	+	-	-	-	-	-	-
Additive model (sum of a periodic deterministic function and a stationary random process) [9,10]	+	+/-	+/-	+	-	-	-

Table 1. Cont.

	Properties of Cyclic Signals						
	One-Dimensional Cyclicity of Signals	Multidimensional Cyclicity of Signals	Two or More Cyclicity Attributes	Uncertainty, Inaccuracy, Randomness of Signals	Variability of the Rhythm of Signals	Stochasticity of the Rhythm Variability	Rhythmic Connection between Several Cyclic Signals
Multiplicative model (product of a periodic deterministic function and a stationary random process) [11]	+	+/-	+/-	+	-	-	-
Additive–multiplicative model (combines additive and multiplicative models into a single model) [12]	+	+/-	+/-	+	-	-	-
Process with independent (uncorrelated) periodic values (periodic white noise) [3]	+	+	+/-	+	-	-	-
Process with independent (uncorrelated) periodic increments [3]	+	+	+/-	+	-	-	-
Periodically correlated random processes (wide-sense cyclostationary processes) [13–15]	+	+	+	+	-	-	-
Random process with periodic moment functions of higher order (higher-order cyclostationary process) [16,17]	+	+	+	+	-	-	-
Periodic Markov random process [18–21]	+	+	+	+	-	-	-
Periodic random process (strict-sense cyclostationary process, periodically distributed random process) [3,4,22,23]	+	+	+	+	-	-	-
Periodically correlated random vector [24]	+	+	+	+	-	-	+
Periodic random vector [24]	+	+	+	+	-	-	+
Periodic random field [25]	+	+	+	+	-	-	+/-

“+”—takes into account (reflects), “-”—does not take into account (does not reflect), and “+/-”—takes into account (reflects) partially (only under certain additional conditions).

Table 2. Structural mathematical models of cyclic signals based on the generalization of a periodic patterns in the direction that mainly concerns the expansion of their spectral properties.

	Properties of Cyclic Signals						
	One-Dimensional Cyclicity of Signals	Multidimensional Cyclicity of Signals	Two or More Cyclicity Attributes	Uncertainty, Inaccuracy, Randomness of Signals	Variability of the Rhythm of Signals	Stochasticity of the Rhythm Variability	Rhythmic Connection between Several Cyclic Signals
Poly-periodic function [26]	+/-	-	-	+/-	-	-	-
Almost periodic function [27,28]	+/-	-	-	+/-	-	-	-
Poly-periodic random process [29]	+/-	+/-	+/-	+	-	-	-
Almost periodically correlated random process (almost cyclostationary correlated process) [30]	+/-	+/-	+/-	+	-	-	-
Almost periodic random process (almost cyclostationary process) [24,31]	+/-	+/-	+/-	+	-	-	-
Jointly almost cyclostationary processes [24]	+/-	+/-	+/-	+	-	-	+/-
Generalized almost cyclostationary process [24,32]	+/-	+/-	+/-	+	-	-	-
Spectrally correlated processes [24]	+/-	+/-	+/-	+	-	-	-
Oscillatory almost cyclostationary process [24]	+/-	+/-	+/-	+	+/-	-	-

“+”—takes into account (reflects), “-”—does not take into account (does not reflect), “+/-”—takes into account (reflects) partially (only under certain additional conditions).

Table 3. Structural mathematical models of cyclic signals, which, in a certain way, take into account the irregularity of their rhythm.

	Properties of Cyclic Signals						
	One-Dimensional Cyclicity of Signals	Multidimensional Cyclicity of Signals	Two or More Cyclicity Attributes	Uncertainty, Inaccuracy, Randomness of Signals	Variability of the Rhythm of Signals	Stochasticity of the Rhythm Variability	Rhythmic Connection between Several Cyclic Signals
Quasi-harmonic function [33]	+/-	-	-	-	+/-	-	-
Quasi-periodic function [34,35]	+/-	-	-	-	+/-	-	-
Periodic function with variable period [36]	+/-	-	-	-	+	-	-
Random process with a zone-cyclic structure [1]	+	+	+	+	+/-	-	-
Conditionally periodic random processes with a variable period [36]	+/-	+/-	+/-	+	+	-	-
Time-angle periodically correlated process (angle/time cyclostationary process or cyclo-non-stationary process) [37–39]	+	+	+	+	+	-	-
Irregular cyclostationary process (time-warped cyclostationary process) [40,41]	+	+	+	+	+	-	-
Time-warped almost cyclostationary process [42]	+	+	+	+	+	-	-
Cyclostationary process with evolving period and amplitude [43]	+	+	+	+	+	-	-

“+”—takes into account (reflects), “-”—does not take into account (does not reflect), and “+/-”—takes into account (reflects) partially (only under certain additional conditions).

Table 4. Structural mathematical models of cyclic signals within the framework of the theory of cyclic functional relations.

	Properties of Cyclic Signals						
	One-Dimensional Cyclicity of Signals	Multidimensional Cyclicity of Signals	Two or More Cyclicity Attributes	Uncertainty, Inaccuracy, Randomness of Signals	Variability of the Rhythm of Signals	Stochasticity of the Rhythm Variability	Rhythmic Connection between Several Cyclic Signals
Cyclic numerical function [1,44]	+	-	-	-	+	-	-
Cyclically correlated random process [1,45]	+	+	+	+	+	-	-
Cyclic random process [1,46,47]	+	+	+	+	+	-	-
Vector of cyclic rhythmically connected random processes [1,48]	+	+	+	+	+	-	+
Conditional cyclic random process [49]	+	+	+	+	+	+	-
Interval cyclic function [1]	+	-	-	+	+	-	-
Fuzzy cyclic function [1]	+	-	-	+	+	-	-
Abstract cyclic functional relation [1]	+	+	+	+	+	+/-	+

“+”—takes into account (reflects), “-”—does not take into account (does not reflect), and “+/-”—takes into account (reflects) partially (only under certain additional conditions).

The structural mathematical models of cyclic signals that generalize deterministic and stochastic periodic functions in the direction that mainly concerns the expansion of the spectral properties of periodic deterministic functions and periodic random processes are presented in Table 2.

Table 3 presents the structural mathematical models of cyclic signals, which, in a certain way, take into account the variability (irregularity) of their rhythm.

Table 4 presents the cyclic functional relations that, in an explicit form, reflect the cyclic structure of signals and have formal means of taking into account the irregularity of their rhythm.

The abstract cyclic functional relation (first introduced and explored in papers [1]) presented in Table 4 is a generalized mathematical model of signals with a cyclic space-time structure. Such a generalized nature of this mathematical object made it possible to explicitly express the intuitive concept of a cyclic signal (cyclic process, cyclic phenomenon and cyclic movement) by formal means, which is common to deterministic, stochastic, fuzzy and interval model approaches. Despite the importance of this mathematical object for the creation of a general theory of structural mathematical models and model-based methods for processing cyclic signals, the construction procedure, possible alternative definitions and fundamental properties of the abstract cyclic functional relation in work [1] has remained almost unexplored. In works [45,46], procedures for constructing partial cases of an abstract cyclic functional relation were developed, namely, the procedure for constructing a cyclically correlated random process and a cyclic (cyclically distributed) random process, which are mathematical models of cyclic stochastic signals. In this article, we will extend the procedures for constructing a cyclically correlated random process and a cyclic (cyclically distributed) random process developed in works [45,46] to a more general case, namely, to the case of an abstract cyclic functional relation (cyclic relative attributes functional relation). Also, the article will present and investigate a number of the fundamental properties of abstract cyclic functional relations and alternative definitions of this class of functions. Such a broad generalization of the results of the works [45,46] in the case of an arbitrary type of cyclic functional relation will provide a more systematic and broad view of the modeling of cyclic signals within the framework of various paradigms of mathematical modeling and will become the basis of a general theory of mathematical modeling and processing of signals with a cyclic space-time structure. In addition, the work is aimed at building a developed taxonomy of subclasses of abstract cyclic functional relations that forms a broad system of classes of potential mathematical models of cyclic signals within the framework of deterministic, stochastic, fuzzy and interval approaches to their modeling.

The work is organized as follows. Section 2 is devoted to the conceptual (informal) foundations of the theory of mathematical modeling and processing with a cyclic space-time structure. Section 3 is devoted to the procedure of construction of an abstract cyclic functional relation. Section 4 presents the cyclic, phase and rhythmic structures of an abstract cyclic functional relation; the theorem on the necessary and sufficient conditions for the rhythm function of an abstract cyclic functional relation is given. Section 5 presents some of the examples of cyclic functional relations as potential mathematical models of cyclic signals within deterministic, stochastic, fuzzy and interval modeling paradigms by means of concretization of the range of values and attributes of cyclicity of an abstract cyclic functional relation. Section 6 is devoted to the method of generating (induction) taxonomies of classes of cyclic functional relations. Section 7 presents a discussion of the main advantages of an abstract cyclic functional relation in comparison with known mathematical models of cyclic signals. Chapter 8 concludes the article, where the main conclusions regarding the results obtained in the article are given.

2. Fundamental Concepts “Cyclic Signal”, “Cycle”, “Cyclicity Attribute”, “Phase” and “Rhythm”

We will briefly define such basic concepts (terms) as “cyclic signal”, “cycle”, “cyclicity attribute”, “phase” and “rhythm”, which together form the core of the conceptual (verbal and intuitive) model of the subject area of the general theory of cyclic signals modeling and processing [1].

By the term “cycle” (from the Greek “κυκλος”—circle), we will understand an ordered set of phases, which constitutes a completed fragment (period, block and circle) in the development and deployment of some phenomenon, process or signal. On the other hand, cycles are segments into which a cyclic signal can be divided; at the same time, they consist of the same ordered set of phases, between which there is a correspondence in attribute(s) and the same type of phase ordering. Then, a cyclic signal (cyclic process and cyclic phenomenon) is such a signal (process and phenomenon) that develops (deploys and evolves) as a sequence of cycles that have, in some sense, similar (identical) properties and structure. Properties that are cyclically repeated in the signal structure (regarding these properties, the signal is cyclic), in the general case, we will call cyclicity attributes or, more briefly, attributes. Such attributes can be the values of the function (for example, for deterministic numerical, vector and matrix periodic functions), when cyclicity is thought of in the sense of the equality of the values of the functional relation for its bijectively connected pairs, which are taken from its different cycles; certain characteristics of the cyclic function, relative to which the equivalence of the values of the cyclic function, are introduced, for example, stochastic equivalence in the strict or wide sense for periodic random processes, equivalence with respect to mathematical expectation and the mixed central moment of the second order for periodically correlated random processes, equivalence of vectors in the sense of the equality of their norms for vector-valued functions periodic with respect to the norm, equivalence of square matrices in the sense of equality of their norms for a functional matrix periodic with respect to the norm, a membership function in the sense of Zade fuzzy sets, etc. Considering the above, it can be stated that the cyclicity of a signal is always cyclicity according to its certain attribute or a set of attributes. Therefore, in the future, we will talk about the cyclicity of the signal by its attribute (or attributes) of cyclicity. Then, a signal that is cyclic by attribute (attributes) is an ordered set of cycles between which there is similarity according to this attribute (attributes) and the same type of phase ordering in them.

The concepts of cycle and cyclic signal are based on the concept of phase. The concept of phase (from the Greek “φασις”—emergence) has different semantic nuances and features. In the general scientific sense, phase is a certain stage in the development and deployment of some phenomenon, process or signal. In a strict mathematical sense, the term “phase” is used to denote the argument of a sine or cosine, but in this case, the term “phase” has a narrowly specialized meaning and refers to the simplest types of cyclic functions, namely, harmonics. The term “phase” acquires a broader mathematical meaning within the framework of the theory of phase space (state space). In the theory of phase space, the term “phase” means any element (point) of phase space that does not take into account the cyclic specificity of the deployment of system states in time, since it describes any, in particular, non-cyclic trajectories of the system’s movement, and also does not directly take into account the ordering of phases in the studied process. To display the cyclic structure of a signal under a phase, it is necessary to understand not only its specific value (an element of the phase space) at a specific moment in time, but also its place within the cycle among the rest of its phases, that is, it is necessary to take into account the ordered nature of the phases in each cycle of a cyclic signal. Considering the fact that a signal can be cyclic with respect to a certain attribute (or set of attributes), the intuitive image of a cyclic signal should include the strict equality of the attribute (attributes) of the same type phases in different cycles of the cyclic signal. Namely, for all those elements of the phase space, which are interpreted as one state of the system and which represent readings of the cyclic signal,

located only in its different cycles, the values of the cyclicity attribute (attributes) must be equal.

The term “rhythm” comes from the Greek “ $\rho\upsilon\theta\mu\omicron\varsigma$ ”, which means measured movement, flow and consistency, and it has many modern interpretations. In general, the following two modern interpretations can be distinguished: on the one hand, as an even alternation of ordered elements, in particular, an ordered alternation of cycles; on the other hand, as a tempo, a pulse, which is a characteristic of cyclic movements in relation to the passage (changes) of time. The first interpretation of rhythm is similar to the above interpretation of the term “cyclic signal”, and therefore, in order to minimize synonymy, we will use the term “rhythm” in another sense, namely, as a tempo, a pulse of a cyclic (oscillatory) process. Namely, by rhythm we will understand a property that is inherent in any cyclic signal and that determines the peculiarities of its deployment in time and/or space, or more precisely, sets the values of time (spatial) intervals between single-phase values of the cyclic signal for all its cycles and phases. The rhythm of the cyclic signal can be stable (regular) or it can be unstable (variable or irregular).

3. Procedure of Construction of an Abstract Cyclic Functional Relation

Having correctly matched the above basic informal concepts of the conceptual model with such formal objects as a set, partition of a set, linearly ordered set, functional relation (function), bijective mapping, relational system and isomorphism of relational systems, we will develop a procedure of the construction of an abstract cyclic functional relation and we will give it an appropriate definition.

Based on works [45,46], let us demonstrate the procedure for building a cyclic relative to a set of attributes' $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ (in certain cases, it is allowed that $K = \infty$) functional relation (other names are abstract cyclic functional relation or abstract cyclic function) $f : \mathbf{W} \rightarrow \mathbf{\Psi}$, which adequately and consistently reflects the cyclic structure of a wide class of signals of different nature and space-time structures. The domain of definition \mathbf{W} of the cyclic functional relation $f : \mathbf{W} \rightarrow \mathbf{\Psi}$ is an ordered discrete set $\mathbf{W} = \mathbf{D} = \{t_{ml} \in \mathbf{R}, m \in \mathbf{Z}, l = \overline{1, L}, L \geq 2\}$ or set of real numbers $\mathbf{W} = \mathbf{R}$. In the case of the discreteness of the domain of definition $\mathbf{W} = \mathbf{D}$, the following type of linear ordering takes place for its elements: $t_{m_1 l_1} < t_{m_2 l_2}$, if $m_2 > m_1$, or if $m_2 = m_1$, and $l_2 > l_1$, in other cases, $t_{m_1 l_1} > t_{m_2 l_2}$ ($m_1, m_2 \in \mathbf{Z}, l_1, l_2 \in \overline{1, L}, 0 < t_{m, l+1} - t_{m, l} < \infty$). The range of values of a cyclic functional relation is some linear space $\mathbf{\Psi}$ ($\langle \mathbf{\Psi}, \mathbf{R}, +, * \rangle$ or $\langle \mathbf{\Psi}, \mathbf{C}, +, * \rangle$) over the field of real or complex numbers, the elements of which can be numbers, fuzzy numbers, vectors, matrices, tensors, intervals, functions, random variables, random vectors, random matrices, random functions and random operators, etc. Since the functional relation $f : \mathbf{W} \rightarrow \mathbf{\Psi}$ is a set of pairs (argument t , value $f(t)$) $f = \{(t, f(t)) : t \in \mathbf{W}\}$, then, in the further presentation of the material, we will also mark it f . Since the range of values $\mathbf{\Psi}$ of function f is an arbitrary linear space, then such a function, by analogy with an abstract function in the sense of Bochner, will be called an abstract function (abstract functional relation) or an abstractly valued function (abstractly valued functional relation).

In order to expand the mathematical means of the formalized description of cyclic signals and generalize the set of possible properties relative to which repetition occurs in the structure of the studied cyclic signal, the concept of set of attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ of cyclicity is introduced. Functions $p_k : f^{n_k} \rightarrow A_k$ maps the n_k -th Cartesian power f^{n_k} into some set A_k , which is the set of possible values of the cyclicity attribute of the signal. Elements of the set A_k can be numbers, vectors, functions, etc., and, therefore, functions $p_k : f^{n_k} \rightarrow A_k$ can be numerical functions, functionals or operators.

In other notations functions $p_k : f^{n_k} \rightarrow A_k$, we will present it as follows: $p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k})))$. In order to exclude non-cyclic functions, we will consider

only such functional relations $p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k})))$ from $\{p_k : f^{t_k} \rightarrow A_k, k = \overline{1, K}\}$, for which exists such a number $T \in \mathbf{W}$, that there are such inequalities:

$$\begin{aligned} & p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k}))) \neq \\ & \neq p_k((t_1 + T, f(t_1 + T)), \dots, (t_{n_k} + T, f(t_{n_k} + T)))_{t_1, \dots, t_k \in \mathbf{W}, k = \overline{1, K}}. \end{aligned} \tag{1}$$

Let us have countable partition $D_W^c = \{W_{c_m}, m \in \mathbf{Z}\}$ of definition domain \mathbf{W} , then, for the elements of partition D_W^c , the following relations are performed [1]:

$$\bigcup_{m \in \mathbf{Z}} W_{c_m} = \mathbf{W}, W_{c_m} \neq \emptyset, W_{c_{m_1}} \cap W_{c_{m_2}} = \emptyset, m_1 \neq m_2, m, m_1, m_2 \in \mathbf{Z}, \tag{2}$$

where $W_{c_m} = [\tilde{t}_m, \tilde{t}_{m+1})$, $m \in \mathbf{Z}$ ($0 < \tilde{t}_{m+1} - \tilde{t}_m < \infty$) in the case $\mathbf{W} = \mathbf{R}$, and set $D_c = \{\tilde{t}_m, m \in \mathbf{Z}\}$ is a subset of \mathbf{R} , elements of which correspond to the moments of the beginning of cycles of a cyclic signal. In the works [45,46], the elements W_{c_m} of partition D_W^c are interpreted as carriers of the relational systems $\langle W_{c_m}, \leq \rangle$ with a binary relation of the linear order \leq , and introduced ordered by m countable family $RS_W^c = \{\langle W_{c_m}, \leq \rangle, m \in \mathbf{Z}\}$ of subrelational systems of the relational system $\langle \mathbf{W}, \leq \rangle$, between which there is an isomorphism with respect to the linear order \leq . For the case when $\mathbf{W} = \mathbf{R}$, Figure 1 in [46] conditionally shows this type of isomorphism.

According to the works [45,46], it is easy to show that by bijective mapping $W \iff f$ from a countable family $RS_W^c = \{\langle W_{c_m}, \leq \rangle, m \in \mathbf{Z}\}$, a countable family $RS_f^c = \{\langle f_{c_m}, \leq_2 \rangle, m \in \mathbf{Z}\}$ of the isomorphic with respect to the binary relation of the linear order \leq_2 subrelational systems $\langle f_{c_m}, \leq_2 \rangle$ of the relational system $\langle f, \leq_2 \rangle$ can be built. The linear order \leq_2 here is generated in $f = \{(t, f(t)) : t \in \mathbf{W}\}$ by the linear order \leq in \mathbf{W} ($\langle \mathbf{W}, \leq \rangle \iff \langle f, \leq_2 \rangle$).

The countable family $RS_f^c = \{\langle f_{c_m}, \leq_2 \rangle, m \in \mathbf{Z}\}$ represents one-dimensional isomorphic structures of f . To display multidimensional (k -dimensional) isomorphic structures of f , let us consider the Cartesian degree $f^k = \{((t_1, f(t_1)), \dots, (t_k, f(t_k))) : t_1, \dots, t_k \in \mathbf{W}\}$ of the k -th order ($k \geq 2$) of f , and consider the bijective mapping $W^k \iff f^k$, which can always be constructed, because any k -dimensional vector $(t_1, \dots, t_k) \in W^k$ corresponds to one and only one k -dimensional vector $((t_1, f(t_1)), \dots, (t_k, f(t_k))) \in f^k$ and vice versa, and for the two different k -dimensional vectors $(t_1, \dots, t_k) \in W^k$ and $(t'_1, \dots, t'_k) \in W^k$, the corresponding two k -dimensional vectors $((t_1, f(t_1)), \dots, (t_k, f(t_k))) \in f^k$ and $((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k))) \in f^k$ are also different, and vice versa. The bijective mapping $W^k \iff f^k$ induces (generates) a linear order in the Cartesian degree f^k itself, which, in this case, can be considered as a carrier of the relational system $\langle f^k, \leq_{2k} \rangle$ with a binary relation of the linear order \leq_{2k} . The ordinal type of f^k coincides with the ordinal type of the set W^k . Namely, for any two k -dimensional vectors $((t_1, f(t_1)), \dots, (t_k, f(t_k))) \in f^k$ and $((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k))) \in f^k$, it is always possible to specify their order: $((t_1, f(t_1)), \dots, (t_k, f(t_k))) \leq_{2k} ((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k)))$ if $t_1 \leq t'_1$ or $((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k))) \leq_{2k} ((t_1, f(t_1)), \dots, (t_k, f(t_k)))$ if $t'_1 \leq t_1$. In the case when $t_1 = t'_1$, we will have such an order: $((t_1, f(t_1)), \dots, (t_k, f(t_k))) \leq_{2k} ((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k)))$ if $t_2 \leq t'_2$ or $((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k))) \leq_{2k} ((t_1, f(t_1)), \dots, (t_k, f(t_k)))$ if $t'_2 \leq t_2$. In general, in the case when $t_i = t'_i$ ($i = \overline{2, k-1}$), we will have such an order: $((t_1, f(t_1)), \dots, (t_k, f(t_k))) \leq_{2k} ((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k)))$ if $t_{i+1} \leq t'_{i+1}$ or $((t'_1, f(t'_1)), \dots, (t'_k, f(t'_k))) \leq_{2k} ((t_1, f(t_1)), \dots, (t_k, f(t_k)))$ if $t'_{i+1} \leq t_{i+1}$. In other words, the bijective mapping $W^k \iff f^k$ is an isomorphism between the relational system $\langle W^k, \leq_{2k-1} \rangle$ and the relational system $\langle f^k, \leq_{2k} \rangle$ with respect to the binary relations of the linear order \leq_{2k-1} and \leq_{2k} ($\langle W^k, \leq_{2k-1} \rangle \iff \langle f^k, \leq_{2k} \rangle$). That is, we will talk about f^k as about a linear ordered Cartesian power by the type of ordering of Cartesian power W^k .

According to the work [46], let us form an ordered by m countable partition $D_{W^k}^c = \{W_{c_m} \times W^{k-1}, m \in \mathbf{Z}\}$ of W^k based on the ordered countable partition $D_W^c = \{W_{c_m}, m \in \mathbf{Z}\}$

of domain W . Due to the linear ordering \leq_{2k-1} of the set W^k , elements $W_{c_m} \times W^{k-1}$ of the partition $D_{W^k}^c$ are also linearly ordered sets. Let us consider the elements $W_{c_m} \times W^{k-1}$ of partition $D_{W^k}^c$ as carriers of the relational systems $\langle W_{c_m} \times W^{k-1}, \leq_{2k-1} \rangle$ with a binary relation of the linear order \leq_{2k-1} . Thus, the partition $D_{W^k}^c$ generates an ordered, by m , countable family $RS_{W^k}^c = \{ \langle W_{c_m} \times W^{k-1}, \leq_{2k-1} \rangle, m \in \mathbf{Z} \}$ of the subrelational systems of the relational system $\langle W^k, \leq_{2k-1} \rangle$, between which there is an isomorphism with respect to the linear order \leq_{2k-1} . For the case when $W = \mathbf{R}$, Figure 2 in [46] conditionally shows this type of isomorphism.

Due to the bijective mapping $W^k \iff f^k$, partition $D_{W^k}^c = \{ W_{c_m} \times W^{k-1}, m \in \mathbf{Z} \}$ of W^k generates an ordered countable partition $D_{f^k}^c = \{ f_{c_m} \times f^{k-1} \subset f^k, m \in \mathbf{Z} \}$ of Cartesian power f^k of the k -th order, where every $f_{c_m} \times f^{k-1}$ is the truncation of the f^k to the set $W_{c_m} \times W^{k-1}$. Namely, each set $W_{c_m} \times W^{k-1}$ matches the $f_{c_m} \times f^{k-1}$, which is its image according to bijective mapping $W^k \iff f^k$. That is, every $f_{c_m} \times f^{k-1}$ is the set of those ordered k -dimensional vectors $\{ ((t_1, f(t_1)), \dots, (t_k, f(t_k))) : (t_1, \dots, t_k) \in W_{c_m} \times W^{k-1} \}$ of the f^k , the argument t_1 of which belongs to W_{c_m} , and the arguments $t_2 \dots, t_k$ of which belongs to W .

Since the Cartesian product f^k is the carrier of $\langle f^k, \leq_{2k} \rangle$, then with its partition $D_{f^k}^c$ it is always possible to connect the countable family $RS_{f^k}^c = \{ \langle f_{c_m} \times f^{k-1}, \leq_{2k} \rangle, m \in \mathbf{Z} \}$ of the subrelational systems of $\langle f^k, \leq_{2k} \rangle$. From the isomorphism between the subrelational systems $\{ \langle W_{c_m} \times W^{k-1}, \leq_{2k-1} \rangle, m \in \mathbf{Z} \}$ with respect to the binary relation of the linear order \leq_{2k-1} due to the isomorphism $\langle W^k, \leq_{2k-1} \rangle \iff \langle f^k, \leq_{2k} \rangle$ follows the isomorphism between the subrelational systems $\{ \langle f_{c_m} \times f^{k-1}, \leq_{2k} \rangle, m \in \mathbf{Z} \}$ with respect to the binary relation of the linear order \leq_{2k} . Namely, for any $m_1, m_2 \in \mathbf{Z}$, the arbitrary subrelational systems $\langle f_{c_{m_1}} \times f^{k-1}, \leq_{2k} \rangle$ and $\langle f_{c_{m_2}} \times f^{k-1}, \leq_{2k} \rangle$ from $RS_{f^k}^c$ are isomorphic with respect to the binary relation of the linear order \leq_{2k} , and for any $m \in \mathbf{Z}$ Cartesian product, $f_{c_m} \times f^{k-1}$ is a Cartesian product, linearly ordered by the type of ordering of its domain $W_{c_m} \times W^{k-1}$.

So, taking into account that mentioned above, it can be argued that there is (1) the isomorphism with respect to the binary relations of the linear order \leq_{2k-1} and \leq_{2k} between the relational systems $\langle W^k, \leq_{2k-1} \rangle$ and $\langle f^k, \leq_{2k} \rangle$; (2) the isomorphism with respect to the binary relation of the linear order \leq_{2k-1} between elements of the countable family $RS_{W^k}^c = \{ \langle W_{c_m} \times W^{k-1}, \leq_{2k-1} \rangle, m \in \mathbf{Z} \}$ of the subrelational systems of the relational system $\langle W^k, \leq_{2k-1} \rangle$; (3) the isomorphism with respect to the binary relation of the linear order \leq_{2k} between the elements of the countable family $RS_{f^k}^c = \{ \langle f_{c_m} \times f^{k-1}, \leq_{2k} \rangle, m \in \mathbf{Z} \}$ of the subrelational systems of the relational system $\langle f^k, \leq_{2k} \rangle$; and (4) the isomorphism with respect to the binary relations of the linear order \leq_{2k-1} and \leq_{2k} between arbitrary pair $W_{c_{m_2}} \times W^{k-1}$ and $f_{c_{m_1}} \times f^{k-1}, m_1, m_2 \in \mathbf{Z}$, taken from the countable partition $D_{W^k}^c = \{ W_{c_m} \times W^{k-1}, m \in \mathbf{Z} \}$ of the Cartesian power W^k and from the countable partition $D_{f^k}^c = \{ f_{c_m} \times f^{k-1} \subset f^k, m \in \mathbf{Z} \}$ of the Cartesian power f^k .

Let us introduce a relational system:

$$\langle \{ f^{1k}, k = \overline{1, K} \}, \{ A_k, k = \overline{1, K} \}, \{ \leq_{2n_k}, k = \overline{1, K} \}, \{ p_k : f^{1k} \rightarrow A_k, k = \overline{1, K} \} \rangle, \quad (3)$$

where $\{ f^{1k}, k = \overline{1, K} \}, \{ A_k, k = \overline{1, K} \}$ are sets of carriers and $\{ \leq_{2n_k}, k = \overline{1, K} \}, \{ p_k : f^{1k} \rightarrow A_k, k = \overline{1, K} \}$ are sets of the relations of the relational system (3).

The partition $D_{f^{i_k}}^c = \{f_{c_m} \times f^{i_k-1} \subset f^{i_k}, m \in \mathbf{Z}\}$ of the Cartesian power f^{i_k} of f generates the family of subrelational systems.

$RS_{f^{i_1}, \dots, f^{i_k}}^c = \{ \langle \{f_{c_m} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle, m \in \mathbf{Z} \}$ of the relational system (3), where $\{f_{c_m} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}$ are carriers of the subrelational system $\langle \{f_{c_m} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$. In the case when $n_k = 1$, in Formula (3), assume that $f_{c_m} \times f^0 = f_{c_m}$.

Let us amplify the isomorphism between the relational systems of the family $RS_{f^{i_1}, \dots, f^{i_k}}^c$ by adding the requirements of the equality of values of functions $p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k})))$ for the bijective connected vectors $((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k}))) \in f_{c_{m_1}} \times f^{i_k-1}$ and $((t'_1, f(t'_1)), \dots, (t'_{n_k}, f(t'_{n_k}))) \in f_{c_{m_2}} \times f^{i_k-1}$ from two different arbitrary Cartesian products, $f_{c_{m_1}} \times f^{i_k-1}$ and $f_{c_{m_2}} \times f^{i_k-1}$. Namely, the isomorphism with respect to the binary relations of the linear order $\{ \leq_{2n_k}, k = \overline{1, K}\}$, for the arbitrary two relational systems $\langle \{f_{c_{m_1}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ and $\langle \{f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$, must be supplemented by an isomorphism between them with respect to functional relations $\{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\}$.

This kind of isomorphism between the relational systems $\langle \{f_{c_{m_1}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ and $\langle \{f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ will be called an isomorphism with respect to the linear order and to the set of attributes $\{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\}$. Let us give a strict definition of this type of isomorphism between the relational systems $\langle \{f_{c_{m_1}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ and $\langle \{f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ for any $m_1, m_2 \in \mathbf{Z}$.

Definition 1. The set of bijective mappings $\{f_{c_{m_1}} \times f^{i_k-1} \iff f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}$ between the appropriate Cartesian products $\{f_{c_{m_1}} \times f^{i_k-1}, k = \overline{1, K}\}$ and $\{f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}$, which are carriers of the relational systems $\langle \{f_{c_{m_1}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ and $\langle \{f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$, will be called the set of isomorphisms with respect to the relations of the linear order $\{ \leq_{2n_k}, k = \overline{1, K}\}$, and with respect to the set of attributes $\{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\}$ between relational systems $\langle \{f_{c_{m_1}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ and $\langle \{f_{c_{m_2}} \times f^{i_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$, if the following statements are true.

1. There are isomorphisms between the relational systems $\langle f_{c_{m_1}} \times f^{i_k-1}, A_k, \{ \leq_{2n_k}, p_k : f^{i_k} \rightarrow A_k \} \rangle$ and $\langle f_{c_{m_2}} \times f^{i_k-1}, A_k, \{ \leq_{2n_k}, p_k : f^{i_k} \rightarrow A_k \} \rangle$ with respect to the relation of the linear order \leq_{2n_k} for any $k = \overline{1, K}$.
2. There are isomorphisms between the relational systems $\langle f_{c_{m_1}} \times f^{i_k-1}, A_k, \{ \leq_{2n_k}, p_k : f^{i_k} \rightarrow A_k \} \rangle$ and $\langle f_{c_{m_2}} \times f^{i_k-1}, A_k, \{ \leq_{2n_k}, p_k : f^{i_k} \rightarrow A_k \} \rangle$ with respect to the set of attributes $\{p_k : f^{i_k} \rightarrow A_k, k = \overline{1, K}\}$, namely, for all the bijective connected vectors $((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k}))) \in f_{c_{m_1}} \times f^{i_k-1}$ and $((t'_1, f(t'_1)), \dots, (t'_{n_k}, f(t'_{n_k}))) \in f_{c_{m_2}} \times f^{i_k-1}$ for

any $k = \overline{1, K}$, there are equal values of the functions $p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k})))$, namely,

$$p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k}))) = p_k((t'_1, f(t'_1)), \dots, (t'_{n_k}, f(t'_{n_k}))), t_1 \in W_{c_{m_1}}, t'_1 \in W_{c_{m_2}}, t_{n_k}, t'_{n_k} \in W, t'_1 \leftrightarrow t_1, \dots, t'_{n_k} \leftrightarrow t_{n_k}, m_1, m_2 \in \mathbf{Z}, k = \overline{1, K}, n_k \in \mathbf{N}. \tag{4}$$

Definition 2. The Cartesian products $f_{c_{m_1}} \times f^{n_k-1}$ and $f_{c_{m_2}} \times f^{n_k-1}$, which are carriers of the isomorphic relational systems $\langle \{f_{c_{m_1}} \times f^{n_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{\leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$ and $\langle \{f_{c_{m_2}} \times f^{n_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{\leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\} \rangle$, will be called **the isomorphic Cartesian products with respect to the relation of linear order \leq_{2n_k} and with respect to the attribute $p_k : f^{n_k} \rightarrow A_k$** or will be called more compact—**the isomorphic Cartesian products**.

The family $RS_{f^{n_1}, \dots, f^{n_K}}^c$ of isomorphic subrelational systems, the carriers of which are the elements of the ordered countable partitions $D_{f^{n_k}}^c$ from sequences $\{D_{f^{n_k}}^c = \{f_{c_m} \times f^{n_k-1} \subset f^{n_k}, m \in \mathbf{Z}, k = \overline{1, K}\}$, constructed above, makes it possible to give the definition of a cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation.

Definition 3. Ordered by the domain of the definition W functional relation $f : W \rightarrow \Psi$ with a range of values, Ψ is a **cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation** (or an **abstract cyclic functional relation** or **abstract cyclic function**), if, for each of its ordered n_k -th Cartesian power f^{n_k} exists the ordered countable partition $D_{f^{n_k}}^c$ from set $\{D_{f^{n_k}}^c = \{f_{c_m} \times f^{n_k-1} \subset f^{n_k}, m \in \mathbf{Z}, k = \overline{1, K}\}$, the elements of which are carriers of isomorphic relational systems $RS_{f^{n_1}, \dots, f^{n_K}}^c = \langle \{f_{c_m} \times f^{n_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{\leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\} \rangle, m \in \mathbf{Z}$ with respect to the relations of linear order $\{\leq_{2n_k}, k = \overline{1, K}\}$ and with respect to the set of attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$.

4. The Cycle, Phase and a Rhythm Structures of Abstract Cyclic Functional Relation

Since the multidimensional cycle and phase structures of an abstract cyclic functional relation are essentially similar to the multidimensional cycle and phase structures of the cyclic random process, which were studied in [46], in this article, we will present only the main results for the abstract cyclic functional relation in a concise form.

Definition 4. The minimal ordered partition $D_{f^{n_k}}^c = \{f_{c_m} \times f^{n_k-1} \subset f^{n_k}, m \in \mathbf{Z}\}$ of the Cartesian product f^{n_k} of the abstract cyclic functional relation $f = \{(t, f(t)) : t \in W\}$ into the isomorphic Cartesian products $f_{c_m} \times f^{n_k-1}$ with respect to the relation of linear order \leq_{2n_k} and with respect to the attribute $p_k : f^{n_k} \rightarrow A_k$, will be called **the partition into n_k -dimensional cycles** of the abstract cyclic functional relation f , and the Cartesian product $f_{c_m} \times f^{n_k-1}$ will be called **the m -th n_k -dimensional cycle** of abstract cyclic functional relation f .

Thus, the cyclic structure of the abstract cyclic functional relation f is given by the set $\{D_{f^{n_k}}^c = \{f_{c_m} \times f^{n_k-1} \subset f^{n_k}, m \in \mathbf{Z}\}, k = \overline{1, K}\}$, elements of which are partitions $D_{f^{n_k}}^c$ into n_k -dimensional cycles $f_{c_m} \times f^{n_k-1}$ of the abstract cyclic functional relation f .

Based on the results obtained above, let us present an abstract cyclic functional relation f and its Cartesian product f^{n_k} through their respective cycles, namely:

$$f = \bigcup_{m \in \mathbf{Z}} f_{c_m} \tag{5}$$

$$f^{n_k} = \bigcup_{m \in \mathbb{Z}} f_{c_m} \times f^{n_k-1}, n_k \in \mathbb{N}, k = \overline{1, K}. \tag{6}$$

Since we require that the range of values of a cyclic functional relation is some linear space Ψ over the field of real or complex numbers, then taking into account the property of linearity an abstract cyclic functional relation f and its Cartesian product f^{n_k} can be given in another form, namely:

$$f(t) = \sum_{m \in \mathbb{Z}} \tilde{f}_m(t), t \in W, \tag{7}$$

where $\tilde{f}_m(t)$ in the areas W_{c_m} coincides with f_{c_m} , but in the areas $W \setminus W_{c_m}$, the functional relation $\tilde{f}_m(t)$ is identically equal to zero in the linear space Ψ ($\tilde{f}_m(t) = 0, t \in W \setminus W_{c_m}$).

Similarly to the representations of the abstract cyclic functional relation f and its Cartesian product f^{n_k} according to formulas (5) and (6) can be given representations of the attribute of cyclicity $p_k : f^{n_k} \rightarrow A_k$ (in another designation, $p_k = \{((t_1, \dots, t_{n_k}), p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k})))) : (t_1, \dots, t_{n_k}) \in W^{n_k}\}$) of the functional relation f :

$$p_k = \bigcup_{m \in \mathbb{Z}} p_{k_{c_m}}, p_{k_{c_m}} \neq \emptyset, p_{k_{c_{m_1}}} \cap p_{k_{c_{m_2}}} = \emptyset, m_1 \neq m_2, m, m_1, m_2 \in \mathbb{Z}, k = \overline{1, K}, n_k \in \mathbb{N}, \tag{8}$$

where $p_{k_{c_m}} = \{((t_1, \dots, t_{n_k}), p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k})))) : (t_1, \dots, t_k) \in W_{c_m} \times W^{n_k-1}\}$ is a n_k -dimensional attribute of m -th n_k -dimensional cycle $f_{c_m} \times f^{n_k-1}$ of an abstract cyclic functional relation f .

Similarly to the approach developed in [46], we will construct the phase structure of the abstract cyclic functional relation f . Let us have the domain $W_{c_0} \times W^{n_k-1}$ of the n_k -dimensional 0-th cycle $f_{c_0} \times f^{n_k-1}$ of an abstract cyclic functional relation f . Due to isomorphism between relational systems $\langle W_{c_0} \times W^{n_k-1}, \leq_{2n_k-1} \rangle$ and $\langle W_{c_m} \times W^{n_k-1}, \leq_{2n_k-1} \rangle$ ($m \in \mathbb{Z}$), for any $(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}) \in W_{c_0} \times W^{n_k-1}$ in the domain $W_{c_m} \times W^{n_k-1}$ of arbitrary n_k -dimensional m -th cycle $f_{c_m} \times f^{n_k-1}$, exists only one element $(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) \in W_{c_m} \times W^{n_k-1}$, which is bijectively connected with the element $(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}) ((t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) \leftrightarrow (t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}))$. Since for an abstract cyclic functional relation f , exists a countable set $D_{f^{n_k}}^c$ of n_k -dimensional cycles, then for any n_k -dimensional vector $(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}) \in W_{c_0} \times W^{n_k-1}$, exists a countable set $W_{\psi_1, \dots, \psi_{n_k}}$ of n_k -dimensional vectors $(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}})$, which are bijectively connected to it. Set $W_{\psi_1, \dots, \psi_{n_k}}$ of all bijectively connected vectors with a vector $(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}})$ is defined as follows:

$$W_{\psi_1, \dots, \psi_{n_k}} = \left\{ (t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) : (t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) \in W_{c_m} \times W^{n_k-1}, (t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) \leftrightarrow (t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}), m \in \mathbb{Z} \right\}, \tag{9}$$

$$(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}), (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}.$$

If vector $(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}})$ runs the all set $W_{c_0} \times W^{n_k-1}$, then we obtain the ordered in the indexes $\psi_1, \dots, \psi_{n_k}$ partition $D_{W^{n_k}}^{ph} = \{W_{\psi_1, \dots, \psi_{n_k}}, (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}\}$ of the domain W^{n_k} of the Cartesian product f^{n_k} of the abstract cyclic functional relation f .

By bijective mapping of elements $W_{\psi_1, \dots, \psi_{n_k}}$ from the partition $D_{W^{n_k}}^{ph}$ into subsets $f_{\psi_1, \dots, \psi_{n_k}}$ of the Cartesian product f^{n_k} ($W_{\psi_1, \dots, \psi_{n_k}} \iff f_{\psi_1, \dots, \psi_{n_k}}$), let us create an uncountable partition $D_{f^{n_k}}^{ph} = \{f_{\psi_1, \dots, \psi_{n_k}}, (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}\}$ of the Cartesian product f^{n_k} of the abstract cyclic functional relation f . According to [46], $f_{\psi_1, \dots, \psi_{n_k}}$ is a countable ordered by m set, defined as follows:

$$f_{\psi_1, \dots, \psi_{n_k}} = \left\{ ((t_m^{\psi_1}, f(t_m^{\psi_1})), \dots, (t_m^{\psi_{n_k}}, f(t_m^{\psi_{n_k}}))) : (t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) \in W_{c_m} \times W^{n_k-1}, (t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}}) \leftrightarrow (t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}), m \in \mathbb{Z} \right\}, (t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}}), (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}. \tag{10}$$

According to [46], $f_{\psi_1, \dots, \psi_{n_k}}$ is a countable set of the n_k -dimensional vectors of the Cartesian product f^{n_k} , among which there are no two vectors belonging to the same n_k -dimensional cycle; that is, among the elements of $f_{\psi_1, \dots, \psi_{n_k}}$, there are no two vectors $\left(\left(t_{m_1}^{\psi_1}, f \left(t_{m_1}^{\psi_1} \right) \right), \dots, \left(t_{m_1}^{\psi_{n_k}}, f \left(t_{m_1}^{\psi_{n_k}} \right) \right) \right)$ where $\left(t_{m_1}^{\psi_1}, \dots, t_{m_1}^{\psi_{n_k}} \right) \in W_{c_{m_1}} \times W^{n_k-1}$ and $\left(\left(t_{m_2}^{\psi_1}, f \left(t_{m_2}^{\psi_1} \right) \right), \dots, \left(t_{m_2}^{\psi_{n_k}}, f \left(t_{m_2}^{\psi_{n_k}} \right) \right) \right)$ where $\left(t_{m_2}^{\psi_1}, \dots, t_{m_2}^{\psi_{n_k}} \right) \in W_{c_{m_2}} \times W^{n_k-1}$ for which $W_{c_{m_1}} = W_{c_{m_2}}$.

Let us give the definition of the n_k -dimensional phase of the cyclic functional relation f .

Definition 5. Ordered by the indexes $\psi_1, \dots, \psi_{n_k}$, the partition $D_{f^{n_k}}^{ph} = \left\{ f_{\psi_1, \dots, \psi_{n_k}}, (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1} \right\}$ of the Cartesian product f^{n_k} is called the partition into sets of n_k -dimensional phases of same type and the set $f_{\psi_1, \dots, \psi_{n_k}}$ is called the set of n_k -dimensional phases of same type of an abstract cyclic functional relation f , if $f_{\psi_1, \dots, \psi_{n_k}}$ are countable sets formed according to (10) and for different elements $\left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right)$ and $\left(\left(t_g^{\psi_1}, f \left(t_g^{\psi_1} \right) \right), \dots, \left(t_g^{\psi_{n_k}}, f \left(t_g^{\psi_{n_k}} \right) \right) \right)$ from $f_{\psi_1, \dots, \psi_{n_k}}$, there is the following such equality of the cyclicity attribute $p_k : f^{n_k} \rightarrow A_k$:

$$p_k \left(\left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right) \right) = p_k \left(\left(\left(t_g^{\psi_1}, f \left(t_g^{\psi_1} \right) \right), \dots, \left(t_g^{\psi_{n_k}}, f \left(t_g^{\psi_{n_k}} \right) \right) \right) \right),$$

$$\left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right) \in W_{c_m} \times W^{n_k-1}, \left(t_g^{\psi_1}, \dots, t_g^{\psi_{n_k}} \right) \in W_{c_g} \times W^{n_k-1},$$

$$t_m^{\psi_1} \leftrightarrow t_g^{\psi_1}, \dots, t_m^{\psi_{n_k}} \leftrightarrow t_g^{\psi_{n_k}}, m, g \in \mathbf{Z}, (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}. \tag{11}$$

Definition 6. The m -th element $\left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right)$ of the set $f_{\psi_1, \dots, \psi_{n_k}} = \left\{ \left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right), m \in \mathbf{Z} \right\}$ is called the n_k -dimensional phase in the n_k -dimensional m -th cycle $f_{c_m} \times f^{n_k-1}$ of an abstract cyclic functional relation f .

Definition 7. The set $A_{\psi_1, \dots, \psi_{n_k}}$, determined according to the expression

$$A_{\psi_1, \dots, \psi_{n_k}} = \left\{ \left(f \left(t_m^{\psi_1} \right), \dots, f \left(t_m^{\psi_{n_k}} \right) \right) : \left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right) \in W_{c_m} \times W^{n_k-1}, \left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right) \leftrightarrow \left(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}} \right), m \in \mathbf{Z} \right\},$$

$$\left(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}} \right), (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}$$

is called the $\psi_1, \dots, \psi_{n_k}$ -set of single-phase values of an abstract cyclic functional relation f .

It should be noted that according to [46], there is not one, but a whole set $\left\{ D_{f^{n_k}}^{c_{\psi_1}}, \psi_1 \in W_{c_0} \right\}$ of possible partitions into n_k -dimensional cycles $f_{c_m} \times f^{n_k-1}$ of an abstract cyclic functional relation f . However, for the abstract cyclic functional relation f exists only one partition $D_{f^{n_k}}^{ph} = \left\{ f_{\psi_1, \dots, \psi_{n_k}}, (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1} \right\}$ into sets of n_k -dimensional phases of same type.

Let us represent the Cartesian product f^k of the abstract cyclic functional relation f through the elements of their phase structures, namely, through the elements of the partition $D_{f^{n_k}}^{ph} = \left\{ f_{\psi_1, \dots, \psi_{n_k}}, (\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1} \right\}$:

$$f^{n_k} = \bigcup_{(\psi_1, \dots, \psi_{n_k}) \in W_{c_0} \times W^{n_k-1}} f_{\psi_1, \dots, \psi_{n_k}}, n_k \in \mathbf{N}, \tag{12}$$

Let us represent of attribute of cyclicity $p_k : \mathcal{F}^{n_k} \rightarrow \mathcal{A}_k$ of the functional relation f :

$$p_k = \bigcup_{(\psi_1, \dots, \psi_{n_k}) \in \mathcal{W}_{c_0} \times \mathcal{W}^{n_k-1}} p_{k, \psi_1, \dots, \psi_{n_k}}, n_k \in \mathbb{N}, k = \overline{1, K}. \tag{13}$$

where

$$p_{k, \psi_1, \dots, \psi_{n_k}} = \left\{ \left(\left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right), p_k \left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right) \right) : \left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right) \in \mathcal{W}_{c_m} \times \mathcal{W}^{n_k-1}, \left(t_m^{\psi_1}, \dots, t_m^{\psi_{n_k}} \right) \leftrightarrow \left(t_0^{\psi_1}, \dots, t_0^{\psi_{n_k}} \right), m \in \mathbb{Z} \right\}$$

is a n_k -dimensional attribute of the set $f_{\psi_1, \dots, \psi_{n_k}}$ of n_k -dimensional phases of the same type of an abstract cyclic functional relation f .

The n_k -dimensional m -th cycle $f_{c_m} \times \mathcal{F}^{n_k-1}$ of an abstract cyclic functional relation f can be presented as follows:

$$f_{c_m} \times \mathcal{F}^{n_k-1} = \bigcup_{(\psi_1, \dots, \psi_{n_k}) \in \mathcal{W}_{c_0} \times \mathcal{W}^{n_k-1}} \left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right), m \in \mathbb{Z},$$

or

$$f_{c_m} \times \mathcal{F}^{n_k-1} = \left\{ \left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right) : (\psi_1, \dots, \psi_{n_k}) \in \mathcal{W}_{c_0} \times \mathcal{W}^{n_k-1} \right\}, m \in \mathbb{Z}.$$

Let us represent the Cartesian degree \mathcal{F}^{n_k} through the set $\left\{ \left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right) : m \in \mathbb{Z}, (\psi_1, \dots, \psi_{n_k}) \in \mathcal{W}_{c_0} \times \mathcal{W}^{n_k-1} \right\}$:

$$\begin{aligned} \mathcal{F}^{n_k} &= \bigcup_{m \in \mathbb{Z}} f_{c_m} \times \mathcal{F}^{n_k-1} = \\ &= \bigcup_{m \in \mathbb{Z}} \bigcup_{(\psi_1, \dots, \psi_{n_k}) \in \mathcal{W}_{c_0} \times \mathcal{W}^{n_k-1}} \left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right). \end{aligned} \tag{14}$$

Let us represent the set of n_k -dimensional phases of same type of an abstract cyclic functional relation f :

$$f_{\psi_1, \dots, \psi_{n_k}} = \bigcup_{m \in \mathbb{Z}} \left(\left(t_m^{\psi_1}, f \left(t_m^{\psi_1} \right) \right), \dots, \left(t_m^{\psi_{n_k}}, f \left(t_m^{\psi_{n_k}} \right) \right) \right), (\psi_1, \dots, \psi_{n_k}) \in \mathcal{W}_{c_0} \times \mathcal{W}^{n_k-1}. \tag{15}$$

Based on the results of works [1,45,46], let us formulate and prove the following theorem.

Theorem 1. For the cyclic, by attributes $\{p_k : \mathcal{F}^{n_k} \rightarrow \mathcal{A}_k, k = \overline{1, K}\}$, functional relation $f = \{(t, f(t)) : t \in \mathcal{W}\}$, there exists a numerical function $T(t, n), t \in \mathcal{W}, n \in \mathbb{Z}$, for which the following properties occur:

- (1)

$$\begin{aligned} T(t, n) &> 0 \ (T(t, 1) < \infty), t \in \mathcal{W}, \text{ if } n > 0, \\ T(t, n) &= 0, t \in \mathcal{W}, \text{ if } n = 0, \\ T(t, n) &< 0, t \in \mathcal{W}, \text{ if } n < 0; \end{aligned} \tag{16}$$

- (2) for any $t_1 \in \mathcal{W}$ and $t_2 \in \mathcal{W}$, for which $t_1 < t_2$, and for function $T(t, n)$ a strict inequality holds:

$$T(t_1, n) + t_1 < T(t_2, n) + t_2, \forall n \in \mathbb{Z}; \tag{17}$$

and for each attribute $p_k : \mathcal{F}^{n_k} \rightarrow \mathcal{A}_k$ from the set of attributes $\{p_k : \mathcal{F}^{n_k} \rightarrow \mathcal{A}_k, k = \overline{1, K}\}$ there is the following equality:

$$\begin{aligned} &p_k \left(\left(t_1, f \left(t_1 \right) \right), \dots, \left(t_{n_k}, f \left(t_{n_k} \right) \right) \right) = \\ &= p_k \left(\left(t_1 + T \left(t_1, n \right), f \left(t_1 + T \left(t_1, n \right) \right) \right), \dots, \left(t_{n_k} + T \left(t_{n_k}, n \right), f \left(t_{n_k} + T \left(t_{n_k}, n \right) \right) \right) \right), \end{aligned} \tag{18}$$

$t_1, \dots, t_{n_k} \in \mathcal{W}, n \in \mathbb{Z}, k = \overline{1, K}, n_k \in \mathbb{N}.$

On the contrary, if for the functional relation $f = \{(t, f(t)) : t \in \mathbf{W}\}$, there exists a numerical function $T(t, n)$, $t \in \mathbf{W}$, $n \in \mathbf{Z}$ with all mentioned above properties (16), (17), and if equalities (18) are hold for any $k = \overline{1, K}$, then it is a cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation (an abstract cyclic functional relation).

Proof of Theorem 1. The first part of the procedure for proving this theorem coincides with the first part of proving a similar theorem for a cyclically correlated random process given in [45]. In order to ensure the integrity of the description of the procedure for proving Theorem 1, we will use the results of [1,45]. \square

According to the definition of the cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation $f = \{(t, f(t)) : t \in \mathbf{W}\}$, any of its two cycles, $f_{c_{m_1}} = \{(t, f(t)) \in f : t \in W_{c_{m_1}}\}$ and $f_{c_{m_2}} = \{(t, f(t)) \in f : t \in W_{c_{m_2}}\}$, are isomorphic with respect to the binary relation of the linear order \leq_2 , and this isomorphism is due to the isomorphism with respect to the binary relation of the linear order \leq of the domains $W_{c_{m_1}}$ and $W_{c_{m_2}}$, which are ordinary sets of real numbers. Between the isomorphic numerical sets $W_{c_{m_1}}$ and $W_{c_{m_2}}$, some additive numeric functions $t_{m_2} = t_{m_1} + T_{m_1 m_2}(t_{m_1}) \in W_{c_{m_2}}$, $t_{m_1} \in W_{c_{m_1}}$ and $t_{m_1} = t_{m_2} + T_{m_2 m_1}(t_{m_2}) \in W_{c_{m_1}}$, $t_{m_2} \in W_{c_{m_2}}$ can always be constructed, namely [45], there is the bijection between $W_{c_{m_1}}$ and $W_{c_{m_2}}$ ($m_1, m_2 \in \mathbf{Z}$), and the same type of linear ordering of sets $W_{c_{m_1}}$ and $W_{c_{m_2}}$ takes place, that is $\forall t_{m_1}, t'_{m_1} \in W_{c_{m_1}}, \exists t_{m_2}, t'_{m_2} \in W_{c_{m_2}}$, that $t_{m_2} = t_{m_1} + T_{m_1 m_2}(t_{m_1})$, $t'_{m_2} = t'_{m_1} + T_{m_1 m_2}(t'_{m_1})$ and there is a strong order relation:

$$t_{m_1} + T_{m_1 m_2}(t_{m_1}) < t'_{m_2} = t'_{m_1} + T_{m_1 m_2}(t'_{m_1}), \text{ if } t_{m_1} < t'_{m_1}, \tag{19}$$

and, vice versa, $\forall t_{m_2}, t'_{m_2} \in W_{c_{m_2}} \exists t_{m_1}, t'_{m_1} \in W_{c_{m_1}}$, that $t_{m_1} = t_{m_2} + T_{m_2 m_1}(t_{m_2})$, $t'_{m_1} = t'_{m_2} + T_{m_2 m_1}(t'_{m_2})$, and there is a strong order relation:

$$t_{m_2} + T_{m_2 m_1}(t_{m_2}) < t'_{m_1} = t'_{m_2} + T_{m_2 m_1}(t'_{m_2}), \text{ if } t_{m_2} < t'_{m_2}, \tag{20}$$

Taking into account the isomorphism between all possible pairs of cycles of the cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation f , we introduce a countable-dimensional matrix of increasing numerical functions which specifies bijective mapping between the domains of its corresponding cycles, i.e., the following matrix:

$$\left\{ t_{m_1} + T_{m_1 m_2}(t_{m_1}) \in W_{c_{m_2}}, t_{m_1} \in W_{c_{m_1}}, m_1, m_2 \in \mathbf{Z} \right\}, \tag{21}$$

moreover, on the diagonal of the functional matrix (21) when $m_1 = m_2 = m \in \mathbf{Z}$ we will have numerical functional relations of identity which are automorphisms with respect to the binary relation of the linear order \leq of the domains W_{c_m} , and at the permutation of the places of indices m_1 and m_2 of the function $t_{m_1} + T_{m_1 m_2}(t_{m_1}) \in W_{c_{m_2}}$, $t_{m_1} \in W_{c_{m_1}}$, we obtain the inverse numerical increasing function $t_{m_2} + T_{m_2 m_1}(t_{m_2}) \in W_{c_{m_1}}$, $t_{m_2} \in W_{c_{m_2}}$.

Entering the notations $m_1 = m$, $m_2 = m + n$, $m, n \in \mathbf{Z}$ and taking them into account in the indices of the elements of the matrix (21), i.e., $t_{m+n} = t_m + T_{m, m+n}(t_m) = t_m + T_m(t_m, n)$, from the matrix (21), we obtain the following countable-dimensional vector of increasing numerical functions from two arguments t_m and n :

$$\{t_m + T_m(t_m, n) \in W_{c_{m+n}}, t_m \in W_{c_m}, m, n \in \mathbf{Z}\}. \tag{22}$$

Note that the increase of these numerical functions takes place in relation to both the argument t_m and the argument n . Each element of the countable-dimensional vector (22) establishes an isomorphism between the domains of the definition of the arbitrary m -th

cycle and $m + n$ -th cycle, which is remote from m -th cycle on n cycles. In addition, for all elements of the countable-dimensional vector (22), there are the following inequalities [1,45]:

$$\begin{cases} T_m(t_m, n) > 0, \text{ if } n > 0, \forall m \in \mathbf{Z}, \\ T_m(t_m, n) = 0, \text{ if } n = 0, \forall m \in \mathbf{Z}, \\ T_m(t_m, n) < 0, \text{ if } n < 0, \forall m \in \mathbf{Z}. \end{cases} \tag{23}$$

The first property ($T_m(t_m, n) > 0, \text{ if } n > 0$) follows from the following facts: $\forall t_m \in \mathbf{W}_{c_m}, m \in \mathbf{Z}$ and $\forall n > 0, \exists t_{m+n} = t_m + T_m(t_m, n) \in \mathbf{W}_{c_{m+n}}$. Moreover, $t_{m+n} > t_m$, whereas $n > 0$, and therefore, $T_m(t_m, n) > t_{m+n} - t_m$ and $T_m(t_m, n) > 0$.

The second property ($T_m(t_m, n) = 0, \text{ if } n = 0$) follows from the fact that $t_{m+0} = t_m \in \mathbf{W}_{c_m}$, since $t_m + T_m(t_m, n) = t_m$ and $T_m(t_m, n) = 0$.

The third property ($T_m(t_m, n) < 0, \text{ if } n < 0$) can be proved similarly to the first: $\forall t_m \in \mathbf{W}_{c_m}, m \in \mathbf{Z}$ and $\forall n < 0, \exists t_{m+n} = t_m + T_m(t_m, n) \in \mathbf{W}_{c_{m+n}}$, and $t_{m+n} < t_m$, whereas $n < 0$ and therefore, $t_m + T_m(t_m, n) < t_m$ and $T_m(t_m, n) < 0$.

Since, for an abstract cyclic functional relation f , there is a set $\{D_f^{c\psi_1} = \{f_{c_m}^{\psi_1}, m \in \mathbf{Z}\}, \psi_1 \in \mathbf{W}_{c_0}\}$ of its possible partitions into one-dimensional cycles, therefore, there exists a set of countable-dimensional vectors (22) corresponding to this partitions. However, since, for the cyclic functional relation f , there exists only one its partition $D_f^{p_h} = \{f_{\psi_1}, \psi_1 \in \mathbf{W}_{c_0}\}$ into sets of one-dimensional phases of same type, therefore, for all possible countable-dimensional vectors (22) corresponding to partitions from set $\{D_f^{c\psi_1}, \psi_1 \in \mathbf{W}_{c_0}\}$, there is the one and only one numerical function $t + T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, which is equal to the ordered union (sum) of the elements of the countable-dimensional vector (22) at a fixed n :

$$\{(t, t + T(t, n)), t \in \mathbf{W}\} = \bigcup_{m \in \mathbf{Z}} \{(t_m, t_m + T_m(t_m, n)), t_m \in \mathbf{W}_{c_{m+n}}\}, n \in \mathbf{Z}. \tag{24}$$

Due to the order of the union (24), the numerical function $t + T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, similarly to the elements of the countable-dimensional vector of functions (22), is an isomorphism with respect to the binary relation of the linear order \leq , and therefore, for it a strict inequality (17) holds, i.e., for any fixed n function $t + T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, there is an increasing numerical function. The requirement of the limited function $T(t, n)$ at $n = 1 (T(t, 1) < \infty)$ necessarily follows from the fact that the duration of one-dimensional cycles is limited, that is, formally reflected in the inequalities $0 < \tilde{t}_{m+1} - \tilde{t}_m < \infty$ when considering the partition $D_W^c = \{\mathbf{W}_{c_m}, m \in \mathbf{Z}\}$. The properties (16) follow from the properties (23) of the components of the vector (22), since the numerical function $t + T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, in fact, is ‘‘stitched’’ from these components.

For the n_k -dimensional vector $\left((t_{m_1}^1, f(t_{m_1}^1)), \dots, (t_{m_{n_k}}^{n_k}, f(t_{m_{n_k}}^{n_k})) \right) \in f_{c_{m_1}} \times \dots \times f_{c_{m_{n_k}}}$ $\left((t_{m_1}^1, \dots, t_{m_{n_k}}^{n_k}) \in \mathbf{W}_{c_{m_1}} \times \dots \times \mathbf{W}_{c_{m_{n_k}}}, m_1, \dots, m_{n_k} \in \mathbf{Z} \right)$, there exists a bijective connected to them, the n_k -dimensional vector $\left((t_{m_1}^1 + T(t_{m_1}^1, n), f(t_{m_1}^1 + T(t_{m_1}^1, n))), \dots, (t_{m_{n_k}}^{n_k} + T(t_{m_{n_k}}^{n_k}, n), f(t_{m_{n_k}}^{n_k} + T(t_{m_{n_k}}^{n_k}, n))) \right) \in f_{c_{m_1+n}} \times \dots \times f_{c_{m_{n_k}+n}}$ $\left((t_{m_1+n}^1, \dots, t_{m_{n_k}+n}^{n_k}) \in \mathbf{W}_{c_{m_1+n}} \times \dots \times \mathbf{W}_{c_{m_{n_k}+n}}, m_1, \dots, m_{n_k}, n \in \mathbf{Z} \right)$, and there are such equalities of attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, namely,

$$\begin{aligned} & p_k \left((t_{m_1}^1, f(t_{m_1}^1)), \dots, (t_{m_{n_k}}^{n_k}, f(t_{m_{n_k}}^{n_k})) \right) = \\ = & p_k \left((t_{m_1}^1 + T(t_{m_1}^1, n), f(t_{m_1}^1 + T(t_{m_1}^1, n))), \dots, (t_{m_{n_k}}^{n_k} + T(t_{m_{n_k}}^{n_k}, n), f(t_{m_{n_k}}^{n_k} + T(t_{m_{n_k}}^{n_k}, n))) \right), \tag{25} \\ & \left(t_{m_1}^1, \dots, t_{m_{n_k}}^{n_k} \right) \in \mathbf{W}_{c_{m_1}} \times \dots \times \mathbf{W}_{c_{m_{n_k}}}, m_1, \dots, m_{n_k}, n \in \mathbf{Z}, k = \overline{1, K}, n_k \in \mathbf{N} \end{aligned}$$

If the vector $(t_{m_1}^1, \dots, t_{m_{n_k}}^{n_k})$ runs through the entire set $W_{c_{m_1}} \times \dots \times W_{c_{m_{n_k}}}$ and the vector (m_1, \dots, m_{n_k}) runs through the entire set Z^{n_k} , then equalities (25) will turn into equalities (18), because

$$\bigcup_{m_1, \dots, m_{n_k} \in Z} W_{c_{m_1}} \times \dots \times W_{c_{m_{n_k}}} = W^{n_k}. \tag{26}$$

It is easy to see that, if for some an abstract functional relation f , there exists a numerical function $T(t, n)$, which satisfies the conditions (16), (17), and equations (18) are hold, then such a functional relation is a cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation, because, in this case, there always exists the sequence $\{D_{f^{n_k}}^c = \{f_{c_m} \times f^{n_k-1} \subset f^{n_k}, m \in Z\}, k = \overline{1, K}\}$, the elements of which are the partitions $D_{f^{n_k}}^c$ into the n_k -dimensional cycles $f_{c_m} \times f^{n_k-1}$ of the cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation f , which are carriers of isomorphic relational systems $RS_{f^{n_1}, \dots, f^{n_K}}^c = \{ \langle \{f_{c_m} \times f^{n_k-1}, k = \overline{1, K}\}, \{A_k, k = \overline{1, K}\}, \{ \leq_{2n_k}, k = \overline{1, K}\}, \{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\} \rangle, m \in Z \}$ with respect to the relations of linear order $\{ \leq_{2n_k}, k = \overline{1, K}\}$ and with respect to the set of attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$.

This concludes the proof of the Theorem 1.

Similar to the results of work [45], the function $T(t, n)$, which is the smallest in modulus ($|T(t, n)| = \min_{\gamma \in N} \{|T_\gamma(t, n)|, \gamma \in N\}$, $t \in W, n \in Z$) among all such functions $\{T_\gamma(t, n), \gamma \in N\}$ which satisfy (16)–(18), is called a **rhythm function of an abstract cyclic functional relation f** .

The rhythm function $T(t, n)$ is a mathematical representation of the law of changing the time intervals between the single-phase values of the signals with a cyclic structure. In more detail, the properties of the rhythm function $T(t, n)$ of a cyclic functional relation are studied in works [1,45,46].

The value of the Theorem 1 is that it provides the sufficient and necessary conditions that the function $T(t, n)$ must satisfy, so that the functional relation f was the abstract cyclic functional relation. The possibility of another way of defining an abstract cyclic functional relation directly follows from the Theorem 1.

Definition 8. The functional relation $f : W \rightarrow \Psi$ is called **the cyclic, by attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, functional relation (or abstract cyclic functional relation or abstract cyclic function)**, if, for it, there exists such a function $T(t, n)$, $t \in W, n \in Z$, which satisfies the conditions of the rhythm function, namely,

(1)

$$\begin{aligned} T(t, n) &> 0 \ (T(t, 1) < \infty), t \in W, \text{ if } n > 0, \\ T(t, n) &= 0, t \in W, \text{ if } n = 0, \\ T(t, n) &< 0, t \in W, \text{ if } n < 0; \end{aligned} \tag{27}$$

(2) For any $t_1 \in W$ and $t_2 \in W$, for which $t_1 < t_2$, and for function $T(t, n)$, a strict inequality holds:

$$T(t_1, n) + t_1 < T(t_2, n) + t_2, \forall n \in Z; \tag{28}$$

(3) The function $T(t, n)$ is the smallest in modulus among all such functions $\{T_\gamma(t, n), \gamma \in N\}$ which satisfy (27), (28), namely,

$$|T(t, n)| = \min_{\gamma \in N} \{|T_\gamma(t, n)|, \gamma \in N\}, t \in W, n \in Z; \tag{29}$$

and for each attribute $p_k : f^{n_k} \rightarrow A_k$ from set $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, there is the following equality:

$$\begin{aligned} &p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k}))) = \\ &= p_k((t_1 + T(t_1, n), f(t_1 + T(t_1, n))), \dots, (t_{n_k} + T(t_{n_k}, n), f(t_{n_k} + T(t_{n_k}, n))))), \end{aligned} \tag{30}$$

$t_1, \dots, t_{n_k} \in W, n \in Z, k = \overline{1, K}, n_k \in N.$

A partial case of an abstract cyclic functional relation, when its rhythm function $T(t, n) = n \cdot T$ ($T = \text{const} > 0$), is a **periodic by attributes** $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, **functional relation** (or **abstract periodic functional relation** or **abstract periodic function** or **abstract cyclic functional relation with regular rhythm**). A partial case of abstract cyclic functional relation, when its rhythm function $T(t, n) \neq n \cdot T$, is a **cyclic, by attributes** $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, **functional relation with an irregular rhythm** (or **abstract cyclic functional relation with irregular rhythm** or **abstract cyclic function with irregular rhythm**).

Most of the practical problems of processing cyclic signals on the basis of their mathematical model in the form of an abstract cyclic function is reduced to the identification (estimating) of a set of attributes of cyclicity $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ and rhythm function $T(t, n)$. In this context, it is possible to conditionally distinguish two types of methods for processing cyclic signals, namely, methods of morphoanalysis of cyclic signals, the purpose of which is to identify (estimate) attributes of cyclicity $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ and methods of analyzing the rhythm of cyclic signals, the purpose of which is to identify (estimate) the rhythm function $T(t, n)$.

5. The Simplest Examples of Deterministic, Stochastic, Fuzzy and Interval Cyclic Functional Relations

Let us consider some of the simplest examples of cyclic functional relations as potential mathematical models of signals with a cyclic space-time structure. We will consider only such cases when set of attributes $\{p_k : f^{n_k} \rightarrow A_k, k = \overline{1, K}\} = \{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$, that is, all cyclicity attributes depend on the values of the cyclic function:

$$p_k((t_1, f(t_1)), \dots, (t_{n_k}, f(t_{n_k}))) = p_k(f(t_1), \dots, f(t_{n_k})). \tag{31}$$

First, consider deterministic cyclic functions. Cyclic deterministic functions are a broad subclass of cyclic functions. These functions can be used as mathematical models of cyclic processes and signals if they have a regular repeating structure, or as models in the first approximation, when uncertainty in the structure of the signals can be neglected.

The simplest representative of cyclic deterministic functions is a cyclic numerical function, which is a generalization of a periodic numerical function. If in Definition 8 of the cyclic functional relation $f : W \rightarrow \Psi$, the range of its values Ψ is the set of real numbers ($\Psi = \mathbf{R}$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : \mathbf{R} \rightarrow \mathbf{R}$ ($n_k = 1, A_1 = \mathbf{R}$), namely, $p_1(f(t)) = f(t)$, then, the definition of a cyclic real-valued function follows from the general definition of an abstract cyclic functional relation. According to work [1], we will give the following definition.

Definition 9. The real-valued function $f(t) \in \mathbf{R}, t \in W$ is called the **cyclic real-valued function**, if, for it, there exists such a function $T(t, n), t \in W, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and the following such equality takes place:

$$f(t) = f(t + T(t, n)), t \in W, n \in \mathbf{Z}. \tag{32}$$

An example of a cyclic real-valued function with a constant rhythm is an arbitrary periodic real-valued function. An example of a cyclic real-valued function with a variable rhythm, that is, when $T(t, n) \neq n \cdot T$, is an angularly modulated harmonic function, provided that the modulation is carried out without changing its phase ordering type in all cycles of the modulated function. A typical example of such cyclic function is a function $f_1(t) = \sin(4t^2 + 10t + 1), t > 0$ with rhythm function $T_1(t, 1) = -t - \frac{5}{4} + \sqrt{t^2 + \frac{5}{2}t + \frac{4\pi + 25}{16}}, t > 0$ (see Figure 3).

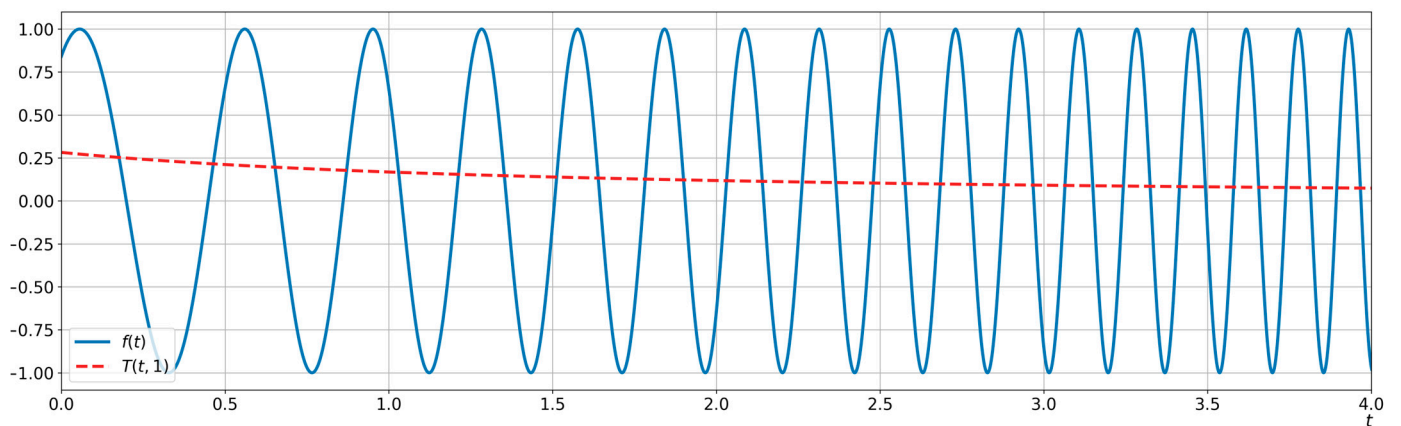


Figure 3. Examples of cyclic real-valued functions and their rhythm functions.

If, in Definition 8 of the cyclic functional relation $f : W \rightarrow \Psi$, the range of its values Ψ is the set of complex numbers ($\Psi = \mathbb{C}$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : \mathbb{C} \rightarrow \mathbb{C}$ ($n_k = 1, A_1 = \mathbb{C}$), namely, $p_1(f(t)) = f(t)$, then, the definition of a cyclic complex-valued function follows from the general definition of an abstract cyclic functional relation. According to work [1], we will give the following definition.

Definition 10. The complex-valued function $f(t) = f_1(t) + i \cdot f_2(t), t \in W$ ($i = \sqrt{-1}$) is called the **cyclic complex-valued function**, if, for, there it exists such a function $T(t, n), t \in W, n \in \mathbb{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and the following such equalities take place:

$$f_1(t) = f_1(t + T(t, n)), t \in W, n \in \mathbb{Z}, \tag{33}$$

$$f_2(t) = f_2(t + T(t, n)), t \in W, n \in \mathbb{Z}. \tag{34}$$

Typical example of such cyclic function is a function $f(t) = \sin(t^2) + i \cdot 2\cos(t^2), t > 0$ with rhythm function $T(t, 1) = -t + \sqrt{t^2 + \pi}, t > 0$ (see Figure 4).

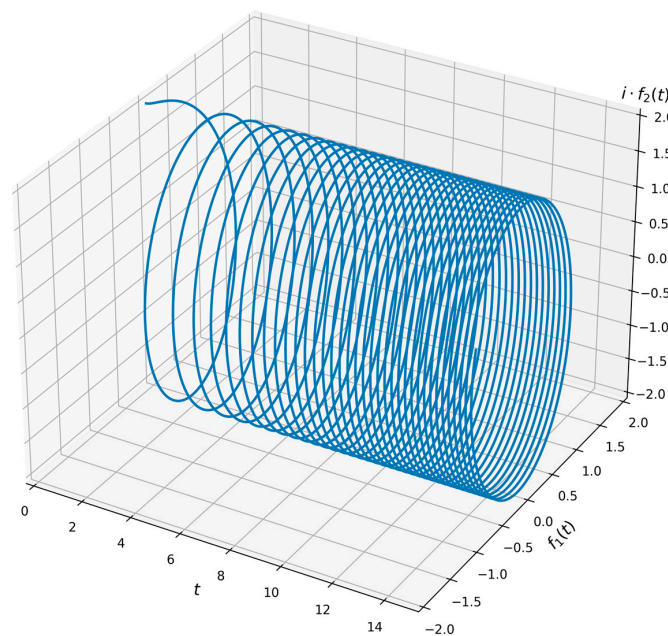


Figure 4. Example of cyclic complex-valued function with a variable rhythm.

If, in Definition 8 of the cyclic functional relation $f : W \rightarrow \Psi$, the range of its values Ψ is the set of complex numbers ($\Psi = \mathbf{C}$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : \mathbf{C} \rightarrow \mathbf{R}$ ($n_k = 1, A_1 = \mathbf{R}$), namely, $p_1(f(t)) = |f(t)| = \sqrt{f_1^2(t) + f_2^2(t)}$, then, the definition of a cyclic, with respect to the modulus, complex-valued function follows from the general definition of an abstract cyclic functional relation [1].

Definition 11. The complex-valued function $f(t) = f_1(t) + i \cdot f_2(t), t \in W$ ($i = \sqrt{-1}$) is called the **cyclic, with respect to the modulus, complex-valued function**, if, for it, there exists such a function $T(t, n), t \in W, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and for the module $|f(t)| = \sqrt{f_1^2(t) + f_2^2(t)}$ the following such equality takes place:

$$|f(t)| = |f(t + T(t, n))|, t \in W, n \in \mathbf{Z}, \tag{35}$$

If, in Definition 8 of the cyclic functional relation $f : W \rightarrow \Psi$, the range of its values Ψ is a vector N -dimensional space ($\Psi = \mathbf{R}^N$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : \mathbf{R}^N \rightarrow \mathbf{R}^N$ ($n_k = 1, A_1 = \mathbf{R}^N$), namely, $p_1(f(t)) = f(t)$, then, the definition of a cyclic vector function $f(t) = \vec{X}(t) = \{x_i(t), i = \overline{1, N}\}$ follows from the general definition of an abstract cyclic functional relation.

Definition 12. The vector function $\vec{X}(t) = \{x_i(t), i = \overline{1, N}\}$ is called the **cyclic vector function (cyclic vector-valued function)**, if for it, there exists such a function $T(t, n), t \in W, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and the following such equalities hold for all components of the vector function $\vec{X}(t)$:

$$x_i(t) = x_i(t + T(t, n)), i = \overline{1, N}, t \in W, n \in \mathbf{Z}. \tag{36}$$

A typical example of a cyclic vector function is a set of frequency-modulated harmonic functions with the same modulation law, provided there is no phase jump of the modulated function. A typical example of such a cyclic function is function $\vec{X}(t) = \{x_i(t), i = \overline{1, 3}\}, t > 0$ with rhythm function $T(t, 1) = -t - \frac{7}{4} + \sqrt{t^2 + \frac{7}{2}t + \frac{8\pi+49}{16}}, t > 0$, where $x_1(t) = \sin(2t^2 + 7t)$ $x_2(t) = \sin(2t^2 + 7t) + 0.5\cos(4t^2 + 14t)$, $x_3(t) = \sin(2t^2 + 7t) + 0.5\cos(4t^2 + 14t) + 0.3\cos(6t^2 + 21t)$ (see Figure 5).

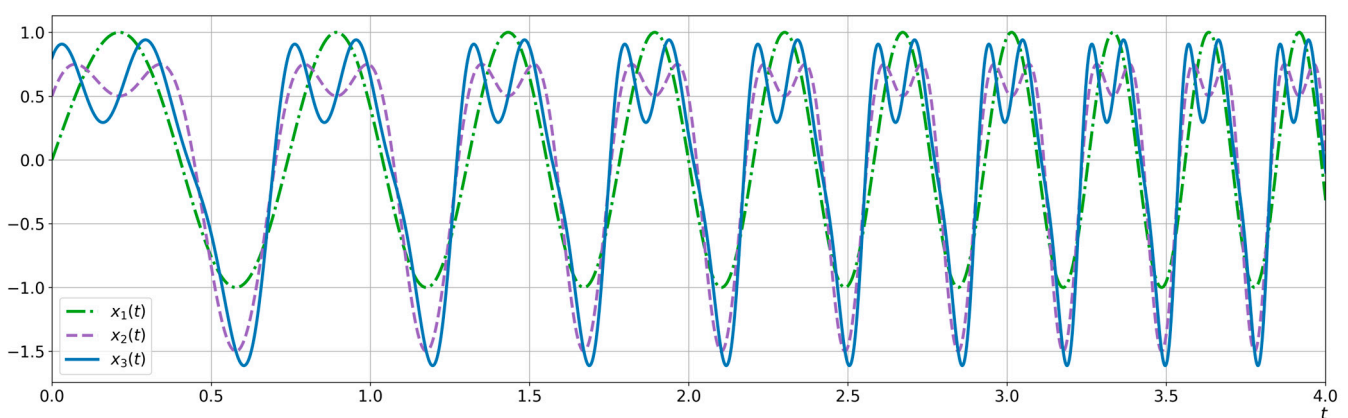


Figure 5. Example of cyclic vector-valued function with a variable rhythm.

If, in Definition 8, of the cyclic functional relation $f : W \rightarrow \Psi$, the range of its values Ψ is a vector N -dimensional space ($\Psi = \mathbf{R}^N$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$

is a set that contains only one function $p_1 : \mathbf{R}^N \rightarrow [0, \infty)$ ($n_k = 1, A_1 = [0, \infty)$), namely, the attribute is the norm $p_1(f(t)) = \|f(t)\|$ in this vector space, then, the definition of a cyclic, with respect to the norm, vector function $f(t) = \vec{X}(t) = \{x_i(t), i = 1, \bar{N}\}$ follows from the general definition of an abstract cyclic functional relation.

Definition 13. The vector function $\vec{X}(t) = \{x_i(t), i = 1, \bar{N}\}$ is called the *cyclic, with respect to the norm, vector function (cyclic, with respect to the norm, vector-valued function)*, if, for it, there exists such a function $T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and equality holds for the norm $\|\vec{X}(t)\|$:

$$\|\vec{X}(t)\| = \|\vec{X}(t + T(t, n))\|, t \in \mathbf{W}, n \in \mathbf{Z}. \tag{37}$$

Depending on which norm of the vector is being considered, it is possible to give the definition of cyclic relative to different norms of the vector function $\vec{X}(t)$. Namely, the cyclicity of the vector function can be introduced, for example, with respect to the quadratic norm $\|\vec{X}(t)\| = \sqrt{\sum_{i=1}^N x_i^2(t)}$, or the norms $\|\vec{X}(t)\| = \max_i \{|x_i(t)|, i = 1, \bar{N}\}$ and $\|\vec{X}(t)\| = \sum_{i=1}^N |x_i(t)|$. It is obvious that any cyclic vector function is also cyclic with respect to its norm, but not vice versa.

If, in Definition 8 of the cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$, the range of its values Ψ is a vector Euclidean N -dimensional space ($\Psi = \mathbf{R}^N$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \bar{1}, \bar{K}\}$ is a set that contains only one function $p_1 : (\mathbf{R}^N)^2 \rightarrow \mathbf{R}$ ($n_1 = 2, A_1 = \mathbf{R}$), namely, the attribute is the scalar product $p_1(f(t_1), f(t_2)) = (f(t_1), f(t_2)) = (\vec{X}(t_1), \vec{X}(t_2)) = \sum_{i=1}^N x_i(t_1) \cdot x_i(t_2) \in \mathbf{R}$ in this Euclidean vector space, then, the definition of a cyclic, with respect to the scalar product $(\vec{X}(t_1), \vec{X}(t_2))$, vector function $f(t) = \vec{X}(t) = \{x_i(t), i = 1, \bar{N}\}$ follows from the general definition of an abstract cyclic functional relation.

Definition 14. The vector function $\vec{X}(t) = \{x_i(t), i = 1, \bar{N}\}$ is called the *cyclic, with respect to the scalar product $(\vec{X}(t_1), \vec{X}(t_2)) = \sum_{i=1}^N x_i(t_1) \cdot x_i(t_2)$, vector function*, if, for it, there exists such a function $T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and equality holds for the scalar product $(\vec{X}(t_1), \vec{X}(t_2))$:

$$(\vec{X}(t_1), \vec{X}(t_2)) = (\vec{X}(t_1 + T(t_1, n)), \vec{X}(t_2 + T(t_2, n))), t_1, t_2 \in \mathbf{W}, n \in \mathbf{Z}. \tag{38}$$

From cyclicity with respect to the scalar product $(\vec{X}(t_1), \vec{X}(t_2)) = \sum_{i=1}^N x_i(t_1) \cdot x_i(t_2)$ vector function $\vec{X}(t)$ followed by its cyclicity relative to the quadratic norm $\|\vec{X}(t)\| = \sqrt{\sum_{i=1}^N x_i^2(t)}$, because $\|\vec{X}(t)\| = \sqrt{(\vec{X}(t), \vec{X}(t))}$. Therefore, in this case, there is cyclicity of the vector function with respect to two attributes, namely, with respect to the attribute $p_1 : \mathbf{R}^N \rightarrow [0, \infty)$ and attribute $p_2 : (\mathbf{R}^N)^2 \rightarrow \mathbf{R}$. It is obvious that any cyclic vector func-

tion is also cyclic with respect to the scalar product, but not vice versa. In particular, the cyclic vector-valued function, the graph of which is given in Figure 3 is cyclic with respect to the scalar product $(\vec{X}(t_1), \vec{X}(t_2)) = \sum_{i=1}^3 x_i(t_1) \cdot x_i(t_2)$ vector function (see Figure 6)

and is cyclic with respect to the quadratic norm $\|\vec{X}(t)\| = \sqrt{\sum_{i=1}^3 x_i^2(t)}$ vector-valued function (see Figure 7) with rhythm function $T(t, 1) = -t - \frac{7}{4} + \sqrt{t^2 + \frac{7}{2}t + \frac{8\pi+49}{16}}$, $t > 0$.

If, in Definition 8 of the cyclic functional relation $f : W \rightarrow \Psi$, the range of its values Ψ is a linear matrix $N \times L$ -dimensional space ($\Psi = M_{N,L}(C)$) (matrix elements are complex numbers), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : M_{N,L}(C) \rightarrow M_{N,L}(C)$ ($n_1 = 1, A_1 = M_{N,L}(C)$), namely, $p_1(f(t)) = f(t)$, then, the definition of a cyclic matrix function (cyclic matrix-valued function) $f(t) = Mat_{N,L}(t) = \{x_{i,j}(t), i = \overline{1, N}, j = \overline{1, L}\}$ follows from the general definition of an abstract cyclic functional relation.

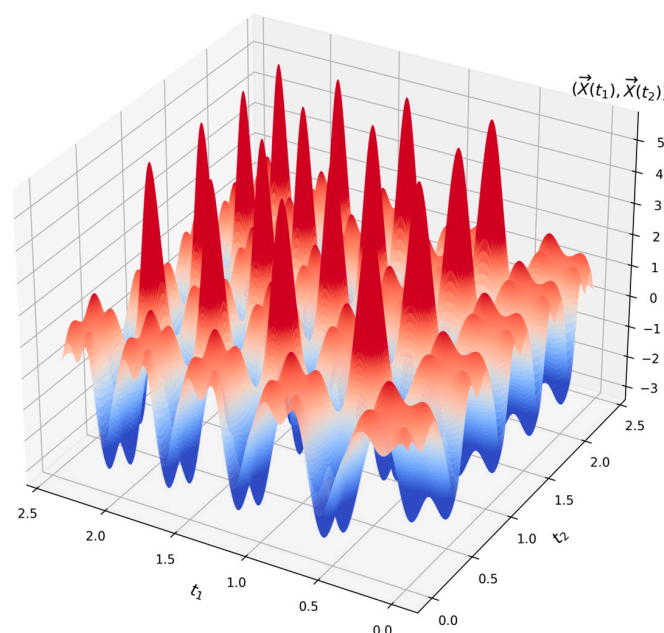


Figure 6. Graphs of scalar product $(\vec{X}(t_1), \vec{X}(t_2)) = \sum_{i=1}^3 x_i(t_1) \cdot x_i(t_2)$ of cyclic vector-valued function $\vec{X}(t) = \{x_i(t), i = \overline{1, 3}\}, t > 0$.

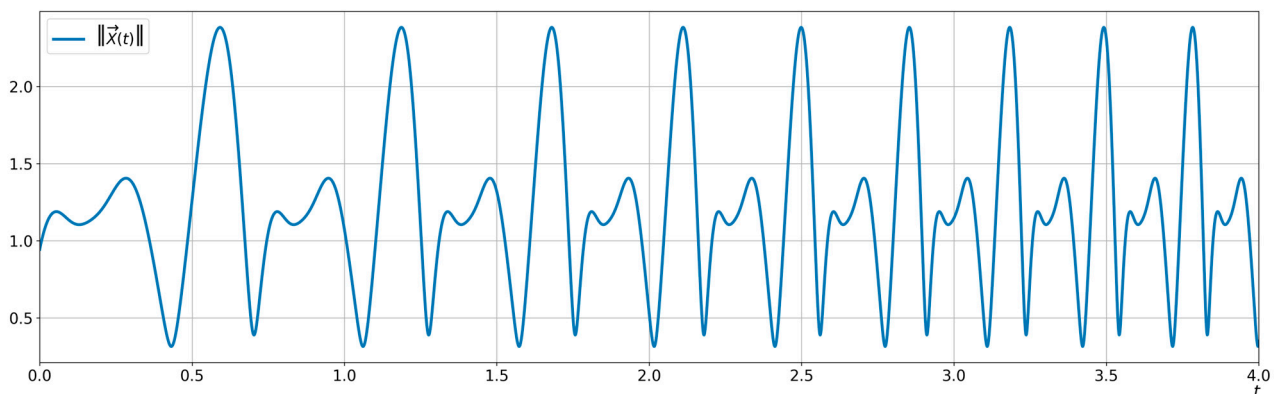


Figure 7. Graphs of quadratic norm $\|\vec{X}(t)\| = \sqrt{\sum_{i=1}^3 x_i^2(t)}$ of cyclic vector-valued function $\vec{X}(t) = \{x_i(t), i = \overline{1, 3}\}, t > 0$.

Definition 15. The matrix function $\mathbf{Mat}_{N,L}(t) = \{x_{i,j}(t), i = \overline{1,N}, j = \overline{1,L}\}$ is called the **cyclic matrix function (or cyclic matrix-valued function)**, if, for it, there exists such a function $T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and the following such equalities hold for all elements of the matrix function $\mathbf{Mat}_{N,L}(t)$:

$$x_{i,j}(t) = x_{i,j}(t + T(t, n)), i = \overline{1,N}, j = \overline{1,L}, t \in \mathbf{W}, n \in \mathbf{Z}. \tag{39}$$

If, in Definition 8 of the cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$, the range of its values Ψ is a linear matrix $N \times L$ -dimensional space ($\Psi = \mathbf{M}_{N,L}(\mathbf{C})$) (matrix elements are complex numbers), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow \mathbf{A}_k, k = \overline{1,K}\}$ is a set that contains only one function $p_1 : \mathbf{M}_{N,L}(\mathbf{C}) \rightarrow [0, \infty)$ ($n_1 = 1, \mathbf{A}_1 = [0, \infty)$), namely, attribute is the norm $p_1(f(t)) = \|f(t)\|$ in this matrix space, then, the definition of a cyclic, with respect to the norm, matrix function $f(t) = \mathbf{Mat}_{N,L}(t) = \{x_{i,j}(t), i = \overline{1,N}, j = \overline{1,L}\}$ follows from the general definition of an abstract cyclic functional relation.

Definition 16. The matrix function $\mathbf{Mat}_{N,L}(t) = \{x_{i,j}(t), i = \overline{1,N}, j = \overline{1,L}\}$ is called the **cyclic, with respect to the norm, matrix function**, if, for it, there exists such a function $T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and equality holds for the norm $\|\mathbf{Mat}_{N,L}(t)\|$:

$$\|\mathbf{Mat}_{N,L}(t)\| = \|\mathbf{Mat}_{N,L}(t + T(t, n))\|, t \in \mathbf{W}, n \in \mathbf{Z}. \tag{40}$$

Depending on which norm of the matrix is considered, it is possible to give a definition of the cyclic relative to different norms of the matrix function $\mathbf{Mat}_{N,L}(t)$. Namely, the cyclicity of the matrix function can be introduced, for example, relative to the quadratic norm $\|\mathbf{Mat}_{N,L}(t)\| = \sqrt{\sum_{i=1}^N \sum_{j=1}^L a_{ij}^2(t)}$, or norms $|\mathbf{Mat}_{N,L}(t)| = \max_{i,j} \{|a_{ij}(t)|, i = \overline{1,N}, j = \overline{1,L}\}$ and $\|\mathbf{Mat}_{N,L}(t)\| = \sum_{i=1}^N \sum_{j=1}^L |a_{ij}(t)|$. It is obvious that any cyclic matrix function is also cyclic with respect to its corresponding norm, but not vice versa.

If, in Definition 8 of the cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$, the range of its values Ψ is a linear functional space $\mathbf{Fsp}_k(\mathbf{C})$, the elements of which are numerical functions of k real variables (for example, three spatial coordinates) ($\Psi = \mathbf{Fsp}_k(\mathbf{C})$) (numerical values of the functions are complex numbers), i.e. $f(x_1, \dots, x_k) \in \Psi, x_1, \dots, x_k \in \mathbf{R}$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow \mathbf{A}_k, k = \overline{1,K}\}$ is a set that contains only one function $p_1 : \mathbf{Fsp}_k(\mathbf{C}) \rightarrow \mathbf{Fsp}_k(\mathbf{C})$ ($n_1 = 1, \mathbf{A}_1 = \mathbf{Fsp}_k(\mathbf{C})$), namely, $p_1(f(t)) = f(t)$, then, the definition of a cyclic field $f(x_1, \dots, x_k, t)$ by argument t follows from the general definition of an abstract cyclic functional relation.

Definition 17. The function $f(x_1, \dots, x_k, t)$ of $k + 1$ real variables is called the **cyclic field (cyclic field by argument t)**, if, for it, there exists such a function $T(t, n), t \in \mathbf{W}, n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and equality takes place:

$$f(x_1, \dots, x_k, t) = f(x_1, \dots, x_k, t + T(t, n)), x_1, \dots, x_k \in \mathbf{R}, t \in \mathbf{W}, n \in \mathbf{Z}. \tag{41}$$

If, in Definition 8 of the cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$, the range of its values Ψ is a linear space $\mathbf{Ten}_{i_1 \dots i_p}(\mathbf{R})$ of tensors (tensor elements are real numbers) of the same dimension ($\Psi = \mathbf{Ten}_{i_1 \dots i_p}(\mathbf{R})$) (the number of indices is the same and the same number of states that the indices i_1, \dots, i_p can acquire), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow \mathbf{A}_k, k = \overline{1,K}\}$ is a set that contains only one function $p_1 : \mathbf{Ten}_{i_1 \dots i_p}(\mathbf{R}) \rightarrow \mathbf{Ten}_{i_1 \dots i_p}(\mathbf{R})$ ($n_1 = 1, \mathbf{A}_1 = \mathbf{Ten}_{i_1 \dots i_p}(\mathbf{R})$), namely, $p_1(f(t)) = f(t)$, then, the definition of a cyclic tensor function (cyclic tensor-valued function) follows from the general definition of an abstract cyclic functional relation.

Definition 18. The tensor function $f(t) = \{a_{i_1 \dots i_p}(t), i_1, \dots, i_p = \overline{1, N}\}$ is called the **cyclic tensor function (or cyclic tensor-valued function)**, if, for it, there exists such a function $T(t, n)$, $t \in W$, $n \in Z$, which satisfies the conditions (27)–(29) of the rhythm function and the following such equalities hold for all elements of the tensor function $f(t)$:

$$a_{i_1 \dots i_p}(t) = a_{i_1 \dots i_p}(t + T(t, n)), i_1, \dots, i_p = \overline{1, N}, t \in W, n \in Z. \tag{42}$$

To represent cyclic random objects, we will denote the cyclic functional relation $f : W \rightarrow \Psi$ as $\zeta : W \rightarrow \Psi$. If, in Definition 8 of the cyclic functional relation $\zeta : W \rightarrow \Psi$, the range of its values Ψ is a space of random variables (for example, the Hilbert space $L_2(\Omega, Prob)$), which are given on the same probability space $(\Omega, F, Prob)$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a family of consistent k -dimensional distribution functions (here $n_k = k$), namely,

$$\begin{aligned} p_k(\zeta(\omega, t_j), j = \overline{1, k}) &= Prob\{\omega : \zeta(\omega, t_1) < x_1, \dots, \zeta(\omega, t_k) < x_k\} = \\ &= F_{k\zeta}^k(x_1, \dots, x_k, t_1, \dots, t_k) \in A_k, x_1, \dots, x_k \in R, t_1, \dots, t_k \in W, \omega \in \Omega, k \in N, \end{aligned} \tag{43}$$

the definition of a cyclic random process follows from the general definition of an abstract cyclic functional relation.

According to works [1,46,47], we will give the definition of a cyclic random process.

Definition 19. The random process $\zeta(\omega, t)$, $\omega \in \Omega$, $t \in W$ is called the **cyclic random process (cyclically distributed random process)**, if, for it, there exists such a function $T(t, n)$, $t \in W$, $n \in Z$, which satisfies the conditions (27)–(29) of the rhythm function and for k -dimensional distribution function $F_{\zeta}^k(x_1, \dots, x_k, t_1, \dots, t_k)$ from the family of consistent distribution functions of a cyclic random process $\zeta(\omega, t)$, $\omega \in \Omega$, $t \in W$ there are there following equalities:

$$\begin{aligned} F_{\zeta}^k(x_1, \dots, x_k, t_1, \dots, t_k) &= F_{\zeta}^k(x_1, \dots, x_k, t_1 + T(t_1, n), \dots, t_k + T(t_k, n)), \\ x_1, \dots, x_k \in R, t_1, \dots, t_k \in W, n \in Z, k \in N \end{aligned} \tag{44}$$

If we consider the cyclicity of a random process within the framework of the correlation theory, namely, if, in Definition 8 of an abstract cyclic functional relation $\zeta : W \rightarrow \Psi$, the range of its values Ψ is a space of random variables (for example, the Hilbert space $L_2(\Omega, Prob)$), which are given on the same probability space $(\Omega, F, Prob)$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a two-element set $\{p_1 : \Psi \rightarrow A_1, p_2 : \Psi^2 \rightarrow A_2\}$, where

$$\begin{aligned} p_1(\zeta(\omega, t)) &= E\{\zeta(\omega, t)\} = m_{\zeta}(t) \in A_1, t \in W, \\ p_2(\zeta(\omega, t_1), \zeta(\omega, t_2)) &= E\{(\zeta(\omega, t_1) - m_{\zeta}(t_1)) \cdot (\zeta(\omega, t_2) - m_{\zeta}(t_2))\} = \\ &= R_{\zeta}^2(t_1, t_2) \in A_2, t_1, t_2 \in W, \omega \in \Omega, \end{aligned} \tag{45}$$

the definition of a cyclically correlated random process follows from the general definition of an abstract cyclic functional relation. Operator $E\{\cdot\}$ is the mathematical expectation operator.

According to the works [1,45], we will give the definition of a cyclically correlated random process.

Definition 20. The random process $\zeta(\omega, t)$, $\omega \in \Omega$, $t \in W$ is called the **cyclically correlated random process**, if, for it, there exists such a function $T(t, n)$, $t \in W$, $n \in Z$, which satisfies the conditions (27)–(29) of the rhythm function and for it mathematical expectation $m_{\zeta}(t)$ and autocovariance function $R_{\zeta}^2(t_1, t_2)$ there are there following equalities:

$$m_{\zeta}(t) = m_{\zeta}(t + T(t, n)), t \in W, n \in Z; \tag{46}$$

$$R_{\zeta}^2(t_1, t_2) = R_{\zeta}^2(t_1 + T(t_1, n), t_2 + T(t_2, n)), t_1, t_2 \in W, n \in Z. \tag{47}$$

If we consider the cyclicity of a random process within the framework of its higher moment functions, namely, if, in Definition 8 of an abstract cyclic functional relation $\zeta : W \rightarrow \Psi$, the range of its values Ψ is a space of random variables (for example, the Hilbert space $L_2(\Omega, Prob)$), which are given on the same probability space $(\Omega, F, Prob)$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : L_2(\Omega, Prob)^k \rightarrow R$ ($n_1 = 1, A_1 = R$), namely,

$$p_1(\zeta(\omega, t_1), \dots, \zeta(\omega, t_k)) = E\left\{\prod_{j=1}^k (\zeta(\omega, t_j))^{r_j}\right\} = C_{\zeta}^p(t_1, \dots, t_k), \tag{48}$$

the definition of a cyclic with respect to the mixed initial moment function $C_{\zeta}^p(t_1, \dots, t_k)$ of order $p = \sum_{j=1}^k r_j$ follows from the general definition of an abstract cyclic functional relation.

Definition 21. The random process $\zeta(\omega, t), \omega \in \Omega, t \in W$ is called the **cyclic with respect to the mixed initial moment function random process**, if, for it, there exists such a function $T(t, n), t \in W, n \in Z$, which satisfies the conditions (27)–(29) of the rhythm function and, for it, the mixed initial moment function $C_{\zeta}^p(t_1, \dots, t_k)$ of order $p = \sum_{j=1}^k r_j$, the following such equality takes place:

$$\begin{aligned} C_{\zeta}^p(t_1, \dots, t_k) &= E\left\{\prod_{j=1}^k (\zeta(\omega, t_j))^{r_j}\right\} = \\ &= C_{\zeta}^p(t_1 + T(t_1, n), \dots, t_k + T(t_k, n)), t_1, \dots, t_k \in W, n \in Z, k \in N. \end{aligned} \tag{49}$$

If, in Definition 8 of an abstract cyclic functional relation $\zeta : W \rightarrow \Psi$, the range of its values Ψ is a space of random variables (for example, the Hilbert space $L_2(\Omega, Prob)$), which are given on the same probability space $(\Omega, F, Prob)$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : L_2(\Omega, Prob)^k \rightarrow R$ ($n_1 = 1, A_1 = R$), namely,

$$p_1(\zeta(\omega, t_1), \dots, \zeta(\omega, t_k)) = E\left\{\prod_{j=1}^k (\zeta(\omega, t_j) - m_{\zeta}(t_j))^{r_j}\right\} = R_{\zeta}^p(t_1, \dots, t_k), \tag{50}$$

the definition of a cyclic, with respect to the mixed central moment, function $R_{\zeta}^p(t_1, \dots, t_k)$ of order $p = \sum_{j=1}^k r_j$ follows from the general definition of an abstract cyclic functional relation.

Definition 22. The random process $\zeta(\omega, t), \omega \in \Omega, t \in W$ is called the **cyclic, with respect to the mixed central moment function, random process**, if, for it, there exists such a function $T(t, n), t \in W, n \in Z$, which satisfies the conditions (27)–(29) of the rhythm function and, for it, the mixed central moment function $R_{\zeta}^p(t_1, \dots, t_k)$ of order $p = \sum_{j=1}^k r_j$, the following such equality takes place:

$$\begin{aligned} R_{\zeta}^p(t_1, \dots, t_k) &= E\left\{\prod_{j=1}^k (\zeta(\omega, t_j) - m_{\zeta}(t_j))^{r_j}\right\} = \\ &= R_{\zeta}^p(t_1 + T(t_1, n), \dots, t_k + T(t_k, n)), t_1, \dots, t_k \in W, n \in Z, k \in N. \end{aligned} \tag{51}$$

Based on the above results, in a similar way, it is possible to define different subclasses of cyclical according to different probabilistic characteristics random processes, namely, according to the domain of definition (discrete and continuous), according to the type of their distribution functions (Gaussian, Poisson, uniformly, exponential distributed, etc.), by belonging to the corresponding class of random processes, the defining feature of which does not contradict the idea of cyclicity (i.e., cyclic white noise, processes with indepen-

dent cyclic increments, linear cyclic random processes, Markov cyclic random processes), by the appearance of the rhythm function (i.e., the rhythm is stable, the rhythm is variable), by probabilistic characteristics in which a cyclic structure is postulated. One of the simplest examples of a cyclic random processes is the random process with a cyclic mathematical expectation. An example of a random process with a cyclic mathematical expectation is the process $\zeta(\omega, t) = \xi_{2\pi}(\omega, t + 0.3 \cdot \sin(0.1 \cdot t))$, where $\xi_{2\pi}(\omega, t) = 2 \cdot \sin(t) + 1.4 \cdot \sin(2t) + \zeta(\omega, t)$, $\omega \in \Omega$, $t > 0$ and $\zeta(\omega, t)$ is a white noise with a normal distribution with zero mathematical expectation and unit variance. The mathematical expectation $m_{\zeta}(t)$ of the random process $\zeta(\omega, t)$ is equal: $m_{\zeta}(t) = E\{\zeta(\omega, t)\} = E\{\xi_{2\pi}(\omega, t + 0.3 \cdot \sin(0.1 \cdot t))\} = E\{\xi_{2\pi}(\omega, t + 0.3 \cdot \sin(0.1 \cdot t)) + \zeta(\omega, t + 0.3 \cdot \sin(0.1 \cdot t))\} = 2 \cdot \sin(t + 0.3 \cdot \sin(0.1 \cdot t)) + 1.4 \cdot \sin(2t + 0.6 \cdot \sin(0.1 \cdot t))$ $t > 0$.

If, in Definition 8 of an abstract cyclic functional relation $\zeta : W \rightarrow \Psi$, the range of its values Ψ is a space of random vectors of dimension N (for example, $(L_2(\Omega, Prob))^N$), which are given on the same probability space $(\Omega, F, Prob)$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a family of consistent k -dimensional distribution functions (here $n_k = k$), namely,

$$p_k(\zeta_1(\omega, t_1), \dots, \zeta_k(\omega, t_k)) = Prob\{\omega : \zeta_1(\omega, t_1) < x_1, \dots, \zeta_k(\omega, t_k) < x_k\} = F_{k_{\zeta_{i_1} \dots \zeta_{i_k}}}(x_1, \dots, x_k; t_1, \dots, t_k) \in A_k, \tag{52}$$

$$i_1, \dots, i_k = \overline{1, N}, x_1, \dots, x_k \in R, t_1, \dots, t_k \in W, \omega \in \Omega, k \in N,$$

the definition of a vector of cyclic rhythmically connected random processes follows from the general definition of an abstract cyclic functional relation.

According to works [1,48], we will give the definition of a vector of cyclic rhythmically connected random processes.

Definition 23. Vector $\Theta_N(\omega, t)$ of random processes $\{\zeta_i(\omega, t), i = \overline{1, N}, \omega \in \Omega, t \in W\}$ is called the **vector of cyclic rhythmically connected random processes** (and the processes themselves is called the **cyclic rhythmically connected random processes**), if, for it, there exists such a function $T(t, n)$, $t \in W$, $n \in Z$, which satisfies the conditions (27)–(29) of the rhythm function and for compatible k -dimensional distribution function $F_{k_{\zeta_{i_1} \dots \zeta_{i_k}}}(x_1, \dots, x_k; t_1, \dots, t_k)$ from the family of consistent distribution functions of a vector $\Theta_N(\omega, t)$ there are the following equalities:

$$F_{k_{\zeta_{i_1} \dots \zeta_{i_k}}}(x_1, \dots, x_k; t_1, \dots, t_k) = F_{k_{\zeta_{i_1} \dots \zeta_{i_k}}}(x_1, \dots, x_k; t_1 + T(t_1, n), \dots, t_k + T(t_k, n)), \tag{53}$$

$$i_1, \dots, i_k = \overline{1, N}, x_1, \dots, x_k \in R, t_1, \dots, t_k \in W, n \in Z, k \in N.$$

If we consider the cyclicity of a random vector within the framework of the correlation theory, namely, if, in Definition 8 of an abstract cyclic functional relation $\zeta : W \rightarrow \Psi$, the range of its values Ψ is a space of random vectors of dimension N (for example, $(L_2(\Omega, Prob))^N$), which are given on the same probability space $(\Omega, F, Prob)$, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a two-element set $\{p_1 : \Psi \rightarrow A_1, p_2 : \Psi^2 \rightarrow A_2\}$, where

$$p_1(\zeta_i(\omega, t)) = E\{\zeta_i(\omega, t)\} = m_{\zeta_i}(t) \in A_1, t \in W, \tag{54}$$

$$p_2(\zeta_{i_1}(\omega, t_1), \zeta_{i_2}(\omega, t_2)) = E\left\{\left(\zeta_{i_1}(\omega, t_1) - m_{\zeta_{i_1}}(t_1)\right) \cdot \left(\zeta_{i_2}(\omega, t_2) - m_{\zeta_{i_2}}(t_2)\right)\right\} = R_{\zeta_{i_1} \zeta_{i_2}}^2(t_1, t_2) \in A_2, t_1, t_2 \in W, \omega \in \Omega, \tag{55}$$

the definition of a vector of cyclically correlated rhythmically connected random processes follows from the general definition of an abstract cyclic functional relation.

Definition 24. Vector $\Theta_N(\omega, t)$ of random processes $\{\zeta_i(\omega, t), i = \overline{1, N}, \omega \in \Omega, t \in W\}$ is called the **vector of cyclically correlated rhythmically connected random processes** (and

the processes themselves is called **the cyclically correlated rhythmically connected random processes**), if, for it, there exists such a function $T(t, n)$, $t \in \mathbf{W}$, $n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and for it mathematical expectations $m_{\tilde{\zeta}_i}(t)$ and autocovariance functions $R_{\tilde{\zeta}_{i_1} \tilde{\zeta}_{i_2}}^2(t_1, t_2)$ there are the following equalities:

$$m_{\tilde{\zeta}_i}(t) = m_{\tilde{\zeta}_i}(t + T(t, n)), t \in \mathbf{W}, n \in \mathbf{Z}, \tag{56}$$

$$R_{\tilde{\zeta}_{i_1} \tilde{\zeta}_{i_2}}^2(t_1, t_2) = R_{\tilde{\zeta}_{i_1} \tilde{\zeta}_{i_2}}^2(t_1 + T(t_1, n), t_2 + T(t_2, n)), i, i_1, i_2 = \overline{1, N}, t_1, t_2 \in \mathbf{W}, n \in \mathbf{Z}. \tag{57}$$

Similarly, it is possible to define the vector of cyclic, with respect to the mixed initial moment, functions $C_{p_{\tilde{\zeta}_{i_1} \dots \tilde{\zeta}_{i_k}}}(t_1, \dots, t_k)$, rhythmically connected random processes and the vector of cyclic, with respect to the mixed central moment functions $R_{p_{\tilde{\zeta}_{i_1} \dots \tilde{\zeta}_{i_k}}}(t_1, \dots, t_k)$, rhythmically connected random processes.

If, in Definition 8 of an abstract cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$, the range of its values Ψ is the set of fuzzy numbers ($\Psi = \mathbf{R}_F$), and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : \mathbf{R}_F \rightarrow A_{I_f}$ ($n_k = 1, A_1 = A_{I_f}, p_1(f(t)) = I_f(x, t) \in A_{I_f}$), where A_{I_f} is a set of membership functions (indicator functions), the definition of a cyclic fuzzy function follows from the general definition of an abstract cyclic functional relation [1].

Definition 25. The fuzzy function $f(t) \in \mathbf{R}_F, t \in \mathbf{W}$ is called the **cyclic fuzzy function**, if, for it, there exists such a function $T(t, n)$, $t \in \mathbf{W}$, $n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and for its membership function $I_f(x, t)$ the following such equality takes place:

$$I_f(x, t) = I_f(x, t + T(t, n)), t \in \mathbf{W}, n \in \mathbf{Z}, 0 \leq I_f(x, t) \leq 1 \tag{58}$$

The cyclic fuzzy function makes it possible to take into account the non-exact repeatability of cyclic signals within the framework of the theory of fuzzy sets in the sense of L. Zade and can be fruitfully applied in the problems of modeling cyclic processes, provided that the oscillatory process is significantly irregular and its number of registered cycles is small, when deterministic and stochastic approaches are inefficient because they lead to low accuracy and reliability of cyclic signal analysis methods under such conditions.

If, in Definition 8 of the cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$ (we will denote the cyclic functional relation $f : \mathbf{W} \rightarrow \Psi$ as $I : \mathbf{W} \rightarrow \Psi$), the range of its values Ψ is the set of open intervals ($\Psi = \mathbf{I}$), that is, the set \mathbf{I} is the set of all possible open intervals, and a set of attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = \overline{1, K}\}$ is a set that contains only one function $p_1 : \mathbf{I} \rightarrow \mathbf{I}$ ($n_k = 1, A_1 = \mathbf{I}$), namely, $p_1(I(t)) = I(t)$, then, the definition of a cyclic interval function (cyclic interval-valued function) follows from the general definition of an abstract cyclic functional relation. According to work [1], we will give the following definition.

Definition 26. The interval-valued function $I(t) \in \mathbf{I}, t \in \mathbf{W}$ is called the **cyclic interval function (cyclic interval-valued function)**, if, for it, there exists such a function $T(t, n)$, $t \in \mathbf{W}$, $n \in \mathbf{Z}$, which satisfies the conditions (27)–(29) of the rhythm function and the following such equality takes place:

$$I(t) = I(t + T(t, n)), t \in \mathbf{W}, n \in \mathbf{Z}. \tag{59}$$

Since a cyclic interval function $I(t)$ can be specified in terms of two cyclic real-valued functions $f_1(t), t \in \mathbf{W}$ and $f_2(t), t \in \mathbf{W}$ ($f_2(t) - f_1(t) = \varepsilon(t) > 0, t \in \mathbf{W}$) with equal rhythm functions $T(t, n)$, namely, $I(t) = (f_1(t), f_2(t)), t \in \mathbf{W}$, then equality (59) is equivalent to the following two such equalities:

$$f_1(t) = f_1(t + T(t, n)) \text{ and } f_2(t) = f_2(t + T(t, n)), t \in \mathbf{W}. \tag{60}$$

With cyclic interval function $I(t)$, it is possible to associate some real-valued function $f(t)$, for which the following such inequality takes place:

$$f(t + T(t, n)) \in (f_1(t + T(t, n)), f_2(t + T(t, n))), t \in W, n \in Z. \tag{61}$$

Function $f(t)$ is not a cyclic function with respect to its values, but it is cyclic with respect to the intervals $(f_1(t), f_2(t))$ from the set I , which can be useful in the problems of modeling (within the deterministic approach) cyclic signals, the single-phase values of which are generally different (not equal) in different cycles, but it is possible to specify the maximum magnitude (upper limit) $\varepsilon(t)$ of the deviation of the single-phase signal values from each other, namely,

$$|f(t) - f(t + T(t, n))| < \varepsilon(t), n \in Z, t \in W. \tag{62}$$

Although property (62) is quite similar to the property of an almost periodic function, in general, these properties are different.

6. Method of Generating (Induction) Taxonomies of Classes of Cyclic Functional Relations as Potential Models of Signals with a Cyclic Structure

The above definitions of various classes of cyclic functional relations are far from exhausting all the possibilities of generating new classes of mathematical models of space-time structure signals and developing their detailed taxonomy. As can be seen from the previously obtained results, and also, as it is shown in works [1,50], by specifying four mathematical objects, namely, the domain of definition W , the domain of values Ψ , the set of attributes of cyclicity $\{p_k : \Psi^{n_k} \rightarrow A_k, k = 1, \bar{K}\}$ and the rhythm function $T(t, n)$ of the abstract cyclic functional relation $f : W \rightarrow \Psi$ can generate less abstract and practically useful classes of cyclic functional relations within the deterministic, stochastic, fuzzy and interval approaches to mathematical modeling of signals with a cyclic structure. In particular, the work [50] proposed a method of generating taxonomies of cyclic functional relations classes, which significantly expands, systematizes, provides a compact form of representation and classifies mathematical models of signals of a cyclic space-time structure, and is also the basis of an axiomatic-deductive strategy for organizing the modern theory of mathematical models and methods of processing cyclic signals. Briefly consider the method of generating taxonomies of cyclic functional relations classes based on an abstract cyclic functional relation.

Let us introduce the following four relational systems, representing four relevant taxonomies:

- (1) $T_\Psi = \{X_\Psi, \subset\}$, where X_Ψ is the carrier of this relational system, which is a set of predetermined classes (types) of linear spaces Ψ in which the corresponding cyclic functional relations $f : W \rightarrow \Psi$ take their values, and \subset is a strict inclusion relation (sets a partial order on the carrier of this relational system);
- (2) $T_A = \{X_A, \subset\}$, where X_A is the carrier of this relational system, which is a set of predetermined classes (types) of possible attributes $\{p_k : \Psi^{n_k} \rightarrow A_k, k = 1, \bar{K}\}$, in which the cyclic structure of the functional relation in $f : W \rightarrow \Psi$ is postulated (displayed);
- (3) $T_{T(t,n)} = \{X_{T(t,n)}, \subset\}$, where $X_{T(t,n)}$ is the carrier of this relational system, which is a set of predetermined classes (types) of rhythm functions $T(t, n)$ of cyclic functional relations $f : W \rightarrow \Psi$;
- (4) $T_W = \{X_W, \subset\}$, where X_W is the carrier of this relational system, which is a set of predetermined types of domains of definition of cyclic functional relations $f : W \rightarrow \Psi$.

According to the work [50], the basic approach to generating (induction) set of classes and taxonomies of cyclic functional relations from an abstract cyclic functional relations consists in the following sequence of steps:

- (1) Formation of the sets X_{Ψ} , X_A , $X_{T(t,n)}$, X_W , as well as specifying the names of the elements of these sets;
- (2) Formation of the taxonomies T_{Ψ} , T_A , $T_{T(t,n)}$, T_W in the form of coded taxonomic trees from sets X_{Ψ} , X_A , $X_{T(t,n)}$, X_W , which consists in repeated application of the logical operation of dividing generic concepts into specific concepts;
- (3) Interpretation of Definition 3 (or Definition 8) of an abstract cyclic functional relation as the identically true 4th-place predicate $P(x_1, x_2, x_3, x_4)$, which is given on sets X_{Ψ} , X_A , $X_{T(t,n)}$, X_W ($x_1 \in X_{\Psi}$, $x_2 \in X_A$, $x_3 \in X_{T(t,n)}$, $x_4 \in X_W$), and which takes its values from a set Def_{cf} of all possible definitions of specific subclasses of an abstract cyclic functional relations $f : W \rightarrow \Psi$;
- (4) Deductive formation of the glossary (set Def_{cf}) and taxonomy (T_{cf}) of classes of cyclic functional relations from the basic Definition 3 (or Definition 8) of an abstract cyclic functional relation $f : W \rightarrow \Psi$, and based on predefined sets X_{Ψ} , X_A , $X_{T(t,n)}$, X_W and taxonomies T_{Ψ} , T_A , $T_{T(t,n)}$, T_W (since the predicate $P(x_1, x_2, x_3, x_4)$ is identically true, then for any sets of values x_1, x_2, x_3, x_4 from sets X_{Ψ} , X_A , $X_{T(t,n)}$, X_W , it always turns into a true statement, namely, it turns into a definition of a specific subclass of cyclic functional relations).

In the work [50], taxonomies T_{Ψ} , T_A , $T_{T(t,n)}$, T_W and T_{cf} are implemented by means of machine-interpreted Web Ontology Language (OWL) and using the Protégé ontology editor. The application of the ontological approach enables the effective systematic solution of a whole range of important methodological and technological tasks in the field of modeling and digital processing of cyclic signals, in particular, (1) ensures the unification and standardization of the technology of presenting information (data and knowledge) in this subject area; (2) enables the creation of a high-quality dictionary (glossary) and knowledge base (thesaurus) in the subject area with the properties of completeness, consistency, interpretability and uniformity and (3) enables multiple reuse of knowledge, which significantly simplifies and intensifies the development of digital technologies for cyclic signal processing.

Based on the method of generating taxonomies of cyclic functional relations classes, many different classes of cyclic functional relations and systems of their taxonomies can be formed. Depending on the approach (paradigm) to the mathematical modeling of signals, four large classes (subclasses) of abstract cyclic functions (ACF) can be distinguished, namely, cyclic deterministic functions (CDF), cyclic stochastic functions (CSF), cyclic fuzzy functions (CFF) and cyclic interval functions (CIF). If we distinguish only two most general types of rhythm of cyclic signals, namely, cyclic signals with a constant (regular) rhythm ($T(t, n) = n \cdot T$) and cyclic signals with a variable (irregular) rhythm ($T(t, n) \neq n \cdot T$), then the above classes of cyclic functional relations can be divided into the following subclasses: cyclic deterministic functions with irregular rhythm (CDFIR) and periodic deterministic functions (PDF), cyclic stochastic functions with irregular rhythm (CSFIR) and periodic stochastic functions (PSF), cyclic fuzzy functions with irregular rhythm (CFFIR) and periodic fuzzy functions (PFF), cyclic interval functions with irregular rhythm (CIFIR) and periodic interval functions (PIF) (see Figure 8).

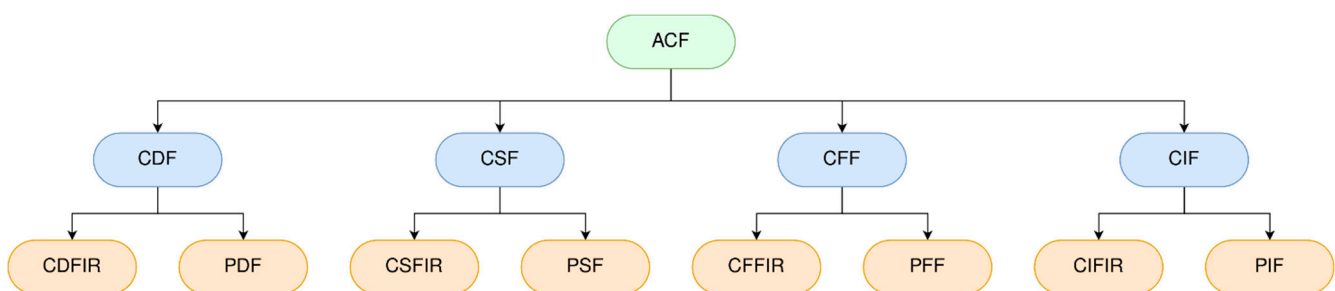


Figure 8. Fragment of the general taxonomy of classes of cyclic functional relations.

Each class presented in Figure 1 is a broad class of functions, which, as its subclasses, includes sets of cyclic functional relations with different types of ranges of values Ψ and attributes of cyclicity $\{p_k : \Psi^{n_k} \rightarrow A_k, k = 1, \bar{K}\}$, which opens up the possibility of theoretically unlimited generation of smaller (more specific) classes of cyclic functional relations within the framework of deterministic, stochastic, fuzzy and interval paradigms of mathematical modeling. Below, we will present only a few small fragments of a possible system of taxonomies of cyclic functional relations in the case of deterministic and stochastic approaches to modeling cyclic signals.

Figure 9 shows a fragment of the taxonomy of cyclic deterministic functions, which includes their subclasses: the class of cyclic real-valued function (CRVF), the class of cyclic complex-valued function (CCVF), the class of cyclic with respect to the modulus real-valued function (CMRVF), the class of cyclic with respect to the modulus complex-valued function (CMCVF), the class of cyclic vector function (CVF), the class of cyclic with respect to the norm vector function (CNVF), the class of cyclic with respect to the scalar product vector function (CSPVF), the class of cyclic matrix function (CMF), the class of cyclic with respect to the norm matrix function (CNMF), the class of cyclic field (CField) and the class of cyclic tensor function (CTF).

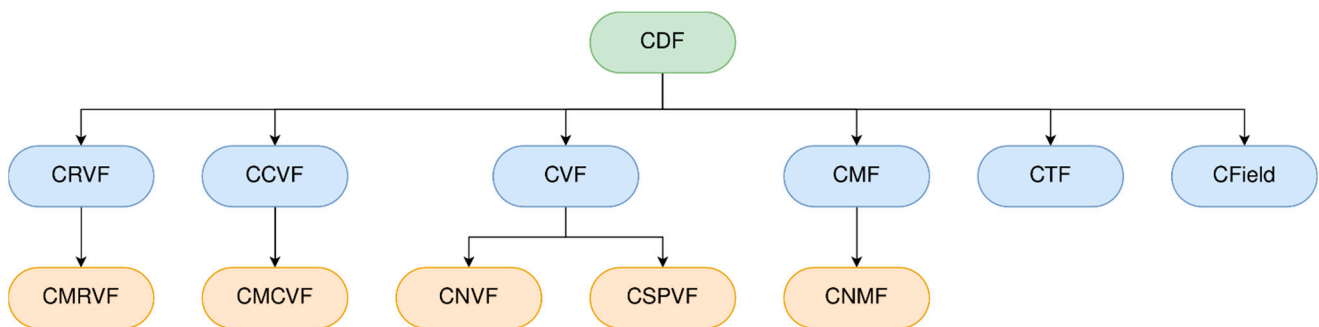


Figure 9. Fragment of the taxonomy of classes of cyclic deterministic functions.

Figure 10 shows a fragment of the taxonomy of cyclic stochastic functions, which includes their subclasses: the class of cyclic random processes (CRP), the class of cyclically correlated random processes (CCRP), the class of periodic random processes (PRP), the class of cyclic random processes with irregular rhythm (CRPIR), the class of cyclic random fields (CRField) and the class of cyclically correlated random fields (CCRField).

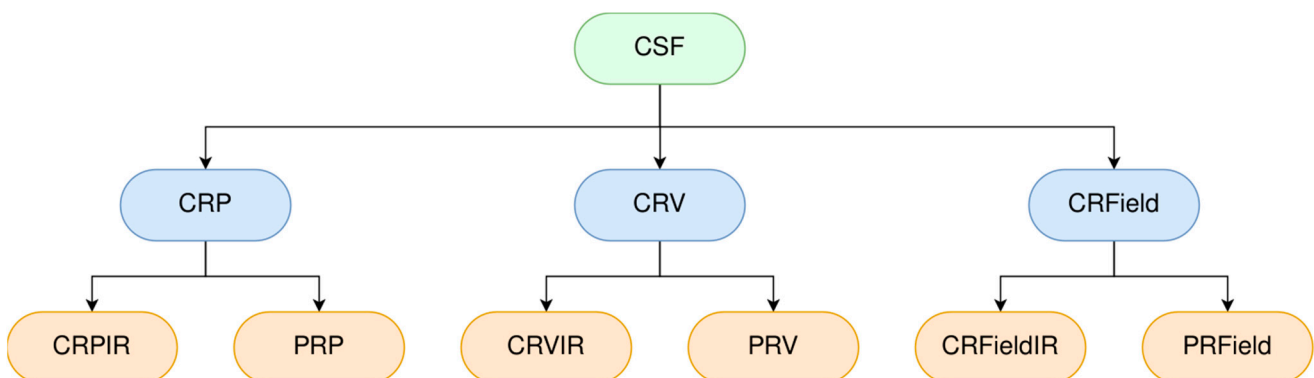


Figure 10. Fragment of the taxonomy of classes of cyclic stochastic functions.

If we consider a class of random processes, some of whose probabilistic characteristics are cyclic functions (let us call such a class of processes the class of random processes with cyclic probabilistic characteristics and denote it as RPCPChar), then, depending on the type of cyclic probabilistic characteristic, it is possible to organize a taxonomy of its subclasses,

which includes such classes random processes: the class of cyclic random processes (CRP), the class of cyclically correlated random processes (CCRP), the class of cyclic with respect to the mixed initial moment function random process (CMIMRP), the class of cyclic with respect to the mixed central moment function random process (CMCMRP), the class of normally distributed cyclic random processes (CNDRP), the class of cyclic random processes with a Poisson distribution (CPDRP), the class of cyclic random processes with a uniform distribution (CUDRP), the class of cyclic Markov random processes (CMRP), the class of cyclic random processes with independent values (class of cyclic white noises) (CWN) and the class of random processes with independent cyclic increments (ICIRP) (see Figure 11).

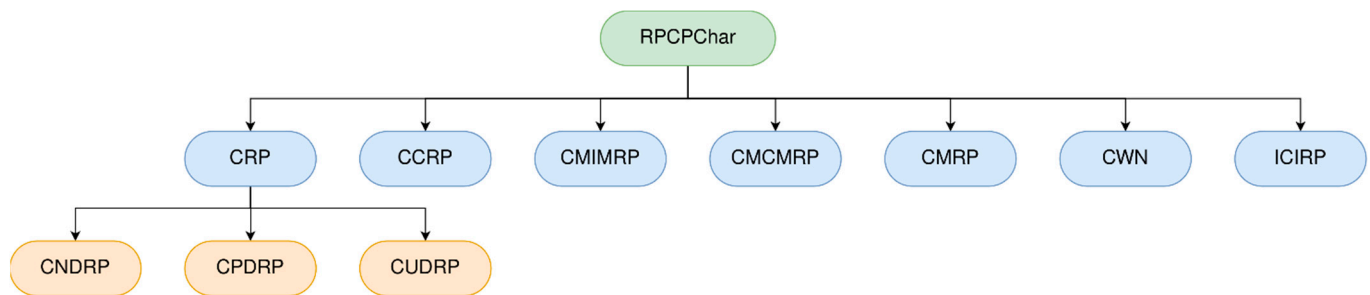


Figure 11. Fragment of the taxonomy of classes of random processes with cyclic probabilistic characteristics.

It is worth noting that, depending on the area of definition of the cyclic functional relation, all the classes shown in Figures 3–6 can be cyclic functional relations of a discrete parameter or cyclic functional relations of a continuous parameter. The fragments of the system of taxonomies of cyclic functional relations constructed above are far from exhausting the entire wealth of classes of cyclic relations and can be significantly expanded and detailed. In particular, one of the possible directions of expanding the system of taxonomies of cyclic functional relations and enriching the class of mathematical models of cyclic signals is the use of not only a deterministic approach to describing the rhythm of a cyclic signal in the form of a deterministic rhythm function, but also the use of stochastic, fuzzy and interval approaches to modeling the rhythm of signals. An example of such an approach, namely, the use of stochastic modeling of the rhythm of cyclic signals, is the construction and use of a conditional cyclic random process [49]. The conditional cyclic random process takes into account the double stochasticity of cyclic signals, namely, the stochasticity of their morphological and rhythmic structures (the rhythm function of such a process is random).

Based on the approach described above, it can be argued that all the theorems, lemmas and conclusions derived from the Definition 3 (or Definition 8) are theorems, lemmas and conclusions for a particular subclass of abstract cyclic functional relation. This approach to the automatic generation of theorems, lemmas and conclusions is described in more detail in the work [50]. The taxonomy of cyclic functional relations is the basis of the theory of mathematical models of cyclic signals and ensures its structure, rigor and formalization, significantly facilitating the identification of new directions and regions of development of new models and methods of processing cyclic signals. Abstract cyclic functional relation and the proposed approach to the organization of the system of taxonomies of cyclic functional relations, from the standpoint of a unified theoretical and methodological approach, takes into account a wide range of possible attributes of cyclicity within the framework of deterministic, stochastic, fuzzy and interval modeling paradigms, a significant structural diversity of patterns of variability and commonality of the rhythm of cyclical signals.

Considering the high level of structuredness and compactness of the system of taxonomies of cyclic functional relations and, taking into account the axiomatic-deductive strategy of building a theory of modeling and processing cyclic signals based on an abstract cyclic functional relation, the development of an onto-oriented expert decision support

system in this field of digital processing is promising. Such an expert system will significantly simplify and automate the procedures for the justified selection of mathematical models and model-based rhythm-adaptive technologies for processing cyclic signals. Such an expert system will be especially useful for developers of digital cyclic signal processing systems who do not have significant experience in the field of mathematical modeling.

7. Discussion

Based on the results obtained above, it can be stated that the main advantages of the abstract cyclic functional relation include the following:

- (1) In contrast to known models of cyclic signals, the abstract cyclic functional relationship in an explicit mathematical form reflects the fundamental patterns of cyclical movement (process, phenomenon) and its cyclic and phase structures, regardless of the type of attributes of cyclicity and the type of rhythm of the investigated signal, which significantly expands the possibilities of applying rigorous mathematical analysis of a wide class of signals with an arbitrary cyclic structure and, to a certain extent, legalizes the use of the term “cyclic function” in the field of mathematics and mathematical modeling;
- (2) The abstract cyclic functional relation as a fundamental and general mathematical model of cyclic signals covers a wide range of attributes of cyclicity (one-dimensional and multidimensional) within the framework of various paradigms of mathematical modeling, which has a significant potential to generalize known deterministic and stochastic mathematical models of cyclic signals and is the basis for the generation of their new mathematical models;
- (3) The abstract cyclic functional relation is the fundamental core of the organization of the theory of mathematical models of cyclic signals, in particular, there is the basis of the deductive generation of specific practically oriented classes of cyclic functional relations and a detailed (broad and deep) system of their taxonomies within the framework of deterministic, stochastic, fuzzy and interval modeling paradigms, which increases the level of systematicity, structuredness, rigor and formalization of the theory modeling and processing of cyclical signals, facilitating the identification of new directions and regions of its development;
- (4) The abstract cyclic functional relation is a flexible and information-rich mathematical object for the generation of a wide class of potential mathematical models, which are the basis for the construction of model-based technologies for digital processing of cyclic signals, which creates grounds for a reasoned choice (in particular with the involvement of expert system technologies), the necessary model and processing method cyclic signals depending on the peculiarities of their morphological and rhythmic structures (they are determined by a set of attributes of cyclicity and the rhythm function) and specific engineering tasks;
- (5) All true theorems, lemmas, conclusions, properties, methods for an abstract cyclic functional relation are also true for any cyclic functional relations belonging to a class from their system of taxonomies \mathbf{T}_{cf} , which significantly simplifies the formation and research of theorems, lemmas, conclusions, properties and methods for arbitrary cyclic functional relations within the framework of deterministic, stochastic, fuzzy and interval approaches to mathematical modeling of cyclic signals;
- (6) In contrast to the known mathematical models of cyclic signals in the form of a periodic deterministic function, a periodically correlated random process, a periodic random process, a periodic random vector and a periodic random field, the abstract cyclic functional relation is a mathematical model of cyclic signals with both stable (regular) and variable (irregular) rhythms, combining these classes of signals into a single class;
- (7) In contrast to mathematical models that take into account the irregularity of the rhythm of cyclic signals, namely, in contrast to a quasi-harmonic function, quasi-periodic function, periodic function with variable period, conditionally periodic ran-

dom processes with variable period, angle/time cyclostationary process, time-warped almost-cyclostationary process, irregular cyclostationary process and cyclostationary process with evolving period and amplitude, the abstract cyclic functional relation has clearly (in an explicit form) given cyclic and phase structures, provides necessary and sufficient conditions for the rhythm function and enables the study of cyclic signals within an arbitrary set of cyclicity attributes for deterministic, stochastic, fuzzy and interval approaches to taking into account the uncertainty of the investigated signals.

It is worth paying additional attention to the differences and commonalities between models of cyclic signals that are based on a periodic pattern (see Table 1), models that generalize periodic deterministic and stochastic functions (see Table 2) and models that are based on a cyclic pattern (see Table 4), which is formally embodied in abstract cyclic functional relation. Despite the fact that the models of cyclic signals that are based on a periodic pattern take into account (implicit in most cases) the cyclic structure of the investigated signals within the framework of deterministic and stochastic paradigms of mathematical modeling, these models lack the ability to reflect the variability (irregularity) of the rhythm (tempo) in the structure of many cyclical signals, which significantly reduces the effectiveness of using such models as the basis of appropriate technologies for the digital processing of cyclic signals with a variable (irregular) rhythm. Models of cyclic signals in the form of cyclic functional relations explicitly reflect the cyclic structure of signals, regardless of the rhythm (regular or irregular) of these signals. It can be expressed more simply as follows: Every periodic pattern is simultaneously cyclic, but not every cyclic pattern is periodic.

The mathematical models presented in Table 2, similar to cyclic functional relations, are certain kinds of generalizations of deterministic and stochastic periodic models of cyclic signals; however, the strategy of such a generalization is fundamentally different from the strategy of generalizing periodic functions within the framework of the theory of cyclic functional relations. Namely, the strategy for constructing cyclic functional relations is based on the explicit representation of the cyclic structure of periodic functions (a sequence of isomorphic segments of a periodic function), which is also preserved for non-periodic cyclic functions with an irregular rhythm, which can be obtained using the action of the scale transformation operator (i.e., time-warping, space-warping) to a periodic function. When constructing and defining the models presented in Table 2, the strategy of generalizing periodic deterministic and stochastic functions does not ensure the preservation of their cyclic structure; instead, the generalization strategy is mainly aimed at expanding their properties in the spectral domain by constructing generalized Fourier series and Fourier transforms, without specifying the conditions for preserving the cyclic structure of the resulting function in the time (space) domain. Such models as poly-periodic function, almost-periodic function, poly-periodic random process, almost-periodically correlated random process, almost-periodic random process, jointly almost-cyclostationary processes, generalized almost-cyclostationary process and spectrally correlated processes, in general, do not have a cyclic structure by themselves (they are not cyclic in the sense of this article), but such a cyclic structure is present in the harmonic components of the representations of these functions using a Fourier series or Fourier transform. In a mathematical model in the form of an oscillatory almost-cyclostationary process, in the general case, there is also no cyclical structure; although, due to its breadth, it includes, as a partial case, the class of cyclically correlated random processes. The spectral components of this model have (not always) a cyclic structure, namely, quasi-harmonic functions (modulated by the angle and phase of the harmonics).

8. Conclusions

In the article, the procedure for construction of an abstract cyclic functional relation was carried out, which summarized the relevant known procedures for a cyclically correlated random process [45] and a cyclic (cyclically distributed) random process [46]. Mathematical definitions of the cyclic and phase structures of the abstract cyclic functional

relation are given, which can be objects of identification (estimation) in most applied problems of cyclic signals processing, and which generalize the cyclic and phase structures of known mathematical models of cyclic signals. A theorem on the invariance of the cyclicity attributes of an abstract cyclic functional relation to shifts (time shifts) of the argument, which are determined by the rhythm function of this functional relation, is formulated and proved. By specifying the range of values and attributes of the cyclicity of an abstract cyclic functional relation, the definitions of important classes of cyclic functional relations (real-valued cyclic numerical function, complex-valued cyclic numerical function, cyclic with respect to the modulus real-valued function, cyclic with respect to the modulus complex-valued function, cyclic vector function, cyclic with respect to the norm vector function, cyclic with respect to the scalar product vector function, cyclic matrix function, cyclic with respect to the norm matrix function, cyclic field, cyclic tensor function, cyclic random process, cyclically correlated random process, cyclic with respect to the mixed initial moment function random process, cyclic with respect to the mixed central moment function random process, vector of cyclic rhythmically connected random processes, vector of cyclically correlated rhythmically connected random processes, cyclic fuzzy function and cyclic interval function) are formulated. A deductive approach to building a wide system of taxonomies of classes of cyclic functional relations as potential mathematical models of cyclic signals is demonstrated, which significantly expands and systematizes mathematical means of describing cyclic signals within the framework of deterministic, stochastic, fuzzy and interval approaches to considering uncertainty in the structure of signals. The results obtained in the article significantly expand and systematize the mathematical tools of the description of cyclic signals and are the basis for the development of effective model-based technologies for processing and computer simulation of signals with a cyclic space-time structure.

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